

**UNIVERSIDADE TÉCNICA DE LISBOA
INSTITUTO SUPERIOR TÉCNICO**

Galois Theory for H-extensions

Marcin Wojciech Szamotulski

Supervisor: Doctor Roger Francis Picken

Co-supervisor: Doctor Christian Edgar Lomp

Thesis approved in public session to obtain the PhD
Degree in Mathematics

Jury final classification: Pass with Merit

Jury

Chairperson: Chairman of the IST Scientific Board

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Doctor Roger Francis Picken

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Funding Institutions

Fundação para a Ciência e a Tecnologia, Portugal

2013

Teoria de Galois para H-extensões

Marcin Szamotulski

Doutoramento em Matemática

Orientador: Prof. Roger Francis Picken

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Resumo

Mostramos que existe uma correspondência de Galois entre subálgebras de uma álgebra de H-comódulo A sobre um anel base R e cocientes generalizados de uma álgebra de Hopf H , se ambos A e H são módulos de Mittag-Leffler chatos. Fornecemos ainda critérios novos para subálgebras e cocientes generalizados serem elementos fechados da conexão de Galois construída. Generalizamos a teoria de objetos admissíveis de Schauenburg a este contexto mais geral. Depois consideramos coextensões de coálgebras de H-módulo. Construimos uma conexão de Galois para elas e provamos que coextensões H-Galois são fechadas. Aplicamos os resultados obtidos à própria álgebra de Hopf, dando uma prova simples que existe uma correspondência biunívoca entre ideais coideais à direita de H e as suas subálgebras de coideal à esquerda, quando H é de dimensão finita. Formulamos ainda uma condição necessária e suficiente para uma correspondência de Galois ser bijetiva quando $A=H$. Consideramos também álgebras de produto cruzado.

Palavras-chave: conexão de Galois, álgebra de Hopf, álgebra de comódulo, coálgebra, biálgebra, reticulado completo, conjunto parcialmente ordenado, elementos fechados, produto cruzado, propriedade de Mittag-Leffler.

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Marcin Szamotulski

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Abstract

We show that there exists a Galois correspondence between subalgebras of an H -comodule algebra A over a base ring R and generalised quotients of a Hopf algebra H if both A and H are flat Mittag-Leffler modules. We also provide new criteria for subalgebras and generalised quotients to be closed elements of the Galois connection constructed. We generalise the Schauenburg theory of admissible objects to this more general context. Then we consider coextensions of H -module coalgebras. We construct a Galois connection for them and we prove that H -Galois coextensions are closed. We apply the results obtained to the Hopf algebra itself and we give a simple proof that there is a bijective correspondence between right ideal coideals of H and its left coideal subalgebras when H is finite dimensional. Furthermore, we formulate a necessary and sufficient condition for a bijective Galois correspondence for $A=H$. We also consider crossed product algebras.

Key-words: Galois connection, Hopf algebra, comodule algebra, coalgebra, bialgebra, complete lattice, partially ordered set, closed elements, crossed product, Mittag-Leffler property, action by monomorphisms.

Acknowledgements

I am deeply thankful to my advisors Professor Roger Picken and Professor Christian Lomp for their guidance and support throughout my research. I also would like to thank IST (Instituto Superior Técnico) and FCT (Fundação para a Ciência e a Tecnologia, grant SFRH/BD/44616/2008) for the financial support essential to successfully complete this thesis.

I would like to thank my friend Dorota Marciniak. Long and deep discussions with her led to the initial development of this subject. She also invited me to study Universal Algebra and Category Theory. Understanding the Universal Algebra approach was essential to solve the stated problems. I also would like to thank my dear parents and my brother for their support and a longtime friend of my mother Heike Ortmann. I would like to thank Stavros Papdakis for his friendship. I would also like to thank Prof. Marian Marciniak whose support and guidance over the years was invaluable.

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Introduction

Hopf–Galois extensions have roots in the approach of [Chase et al. \[1965\]](#) who generalised the *classical Galois Theory* for field extensions to rings which are commutative. [Chase and Sweedler \[1969\]](#) extended these ideas to coactions of Hopf algebras on *commutative algebras* over rings. The general definition of a Hopf–Galois extension was first introduced by [Kreimer and Takeuchi \[1981\]](#). Under the assumption that H is finite dimensional their definition is equivalent to the following now standard one.

Definition

An H -extension $A/A^{co H}$ is called an H -Galois extension if the canonical map of right H -comodules and left A -modules:

$$can : A \otimes_{A^{co H}} A \longrightarrow A \otimes H, \quad a \otimes b \longmapsto ab_{(0)} \otimes b_{(1)} \quad (1)$$

is an isomorphism¹, where $A^{co H} := \{a \in A : a_{(0)} \otimes a_{(1)} = a \otimes 1_H\}$.

A breakthrough in Hopf–Galois theory was made by [van Oystaeyen and Zhang](#) extending the results of [Chase and Sweedler \[1969\]](#) to a noncommutative setting. They construct a Galois correspondence for Hopf–Galois extensions. Van Oystaeyen and Zhang introduced a remarkable construction of an *associated Hopf algebra* to an H -extension $A/A^{co H}$, where A as well as H are supposed to be commutative (see [[van Oystaeyen and Zhang, 1994](#), Sec. 3], for a noncommutative generalisation see: [Schauenburg \[1996, 1998\]](#)). We will denote this Hopf algebra by $L(A, H)$. [Schauenburg \[1998, Prop. 3.2\]](#) generalised the van Oystaeyen and Zhang correspondence (see also [[Schauenburg, 1996](#), Thm 6.4]) to a Galois connection between generalised quotients of the associated Hopf algebra $L(A, H)$ (i.e. quotients by right ideal coideals) and

¹We use the Sweedler notation for coactions and comultiplications: here $\delta : A \longrightarrow A \otimes H$ and for $a \in A$, $\delta(a) \in A \otimes H$ and hence it is a sum of simple tensors $\sum_{k=1}^n a'_k \otimes h_k$ for some $a'_k \in A$ and $h_k \in H$. In Sweedler notation this tensor is denoted as $\sum a_{(0)} \otimes a_{(1)} \in A \otimes H$. We prefer to use the sumless Sweedler notation, e.g. we write $\delta(a) = a_{(0)} \otimes a_{(1)}$, where the suppressed summation symbol is understood.

H -comodule subalgebras of A . We denote the poset of generalised quotients of a Hopf algebra H by $\text{Quot}_{\text{gen}}(H)$. In this work we construct the Galois correspondence:

$$\text{Sub}_{\text{Alg}}(A/B) \begin{matrix} \xrightarrow{\psi} \\ \xleftarrow{\phi} \end{matrix} \text{Quot}_{\text{gen}}(H), \quad \phi(Q) := A^{\text{co}Q} \quad (2)$$

where B is the coinvariants subalgebra of A , without the assumption that B is equal to the commutative base ring R and we also drop the Hopf–Galois assumption (see Theorem 4.2.2 on page 78). We add some module theoretic assumptions on A and H . Instead of the Hopf theoretic approach of van Oystaeyen, Zhang and Schauenburg we propose to look from the lattice theoretic perspective. Using an existence theorem for Galois connections we show that if a comodule algebra A and a Hopf algebra H are flat Mittag–Leffler R -modules (Definition 2.1.29 on page 33) then the Galois correspondence (2) exists. The Mittag–Leffler property appears here since it turns out that a flat R -module M has this property if and only if the endofunctor $M \otimes -$ of the category of left R -modules preserves arbitrary intersections of submodules (see Definition 2.1.26 and Corollary 2.1.36 on page 35). We consider modules with the intersection property in Chapter 2, where we also give examples of flat and faithfully flat modules which fail to have it. For an H -extension $A/A^{\text{co}H}$ over a field we show that $\psi(S) = H/K_S^+H$ where K_S is the smallest left coideal subalgebra of H with the property: $\delta_A(S) \subseteq A \otimes K_S$, where δ_A is the comodule structure map of A (see Theorem 4.2.4 on page 81). Then we discuss Galois closedness of the generalised quotients and subalgebras in (2). We show that a subalgebra $S \in \text{Sub}(A/B)$ is closed if and only if the following canonical map: $\text{can}_S : S \otimes A \rightarrow A \square_{\psi(S)} H$ $\text{can}(s \otimes a) = sa_{(0)} \otimes a_{(1)}$, is an isomorphism (see Theorem 4.3.1 on page 86). We show that if a generalised quotient Q is such that $A/A^{\text{co}Q}$ is Q -Galois then it is necessarily closed under the assumptions that the canonical map of $A/A^{\text{co}H}$ is onto and the unit map $1_A : R \rightarrow A$ is a pure monomorphism of left R -modules (Corollary 4.3.4 on page 88). Later we prove that this is also a necessary condition for Galois closedness if $A = H$ or, more generally, if $A/A^{\text{co}H}$ is a crossed product, H is flat and $A^{\text{co}H}$ is a flat Mittag–Leffler R -module (Theorem 4.7.1 on page 103). For H -Galois extensions over a field we prove that the canonical map $\text{can}_Q : A \otimes_{A^{\text{co}Q}} A \rightarrow A \otimes Q$ is isomorphic if and only if Q is a closed element and the map $\delta_A \otimes \delta_A : A \otimes_{A^{\text{co}Q}} A \rightarrow (A \otimes H) \otimes_{A \otimes H^{\text{co}Q}} (A \otimes H)$ is injective (Theorem 4.6.7 on page 102). We also consider the dual case of H -module coalgebras, which later gives us a simple proof of the bijective correspondence between generalised quotients and left coideal subalgebras of H if it is finite dimensional (Theorem 4.6.1 on page 98):

$$\text{Sub}_{\text{gen}}(H) \begin{matrix} \xrightarrow{\psi} \\ \xleftarrow{\phi} \end{matrix} \text{Quot}_{\text{gen}}(H), \quad \phi(Q) := H^{\text{co}Q}, \quad \psi(K) = H/K^+H \quad (3)$$

[Schauenburg \[1998, Thm. 3.10\]](#) showed that this Galois correspondence restricts to a bijection between admissible quotients and subalgebras, where admissibility assumes, among other things, faithful flatness (for subobjects) and faithful coflatness (for quotient objects) and H is required to be flat over the base ring. The bijectivity of this correspondence, without the assumptions of any faithfully (co)flatness was proved by [Skryabin \[2007\]](#). He showed this by proving that a finite dimensional Hopf algebra is free over any of its left coideal subalgebra. Our independent proof makes no use of this result. We also characterise closed elements of this Galois correspondence for Hopf algebras which are flat over the base ring (Theorem [4.6.2](#) on page [100](#)). As we already mentioned, we show that a generalised quotient Q is closed if and only if H/H^{coQ} is a Q -Galois extension. Furthermore, we note that a left coideal subalgebra K is closed if and only if $H \rightarrow H/K^+H$ is a K -Galois coextension (see Definition [4.5.2](#) on page [96](#)). This gives an answer to the question when a bijective correspondence between generalised quotients over which H is faithfully coflat and coideal subalgebras over which H is faithfully flat extends to a bijective correspondence without (co)flatness assumptions. We also note that for any H -extension A/B over a field a generalised quotient Q which is not of the form H/K^+H cannot be closed.

There are two preprints of mine on the arxiv which are related with this thesis. These are both listed as co-authored with Dorota Marciniak, since we started investigating this subject together, but she was not much involved in the subsequent developments and went on to do her PhD in game theory. The first preprint ([Galois Theory of Hopf Galois Extensions](#), [arXiv:0912.0291](#)) is about Hopf-Galois extensions over fields, and does not include the generalisation of Schauenburg's correspondence between admissible objects. The more recent preprint ([Galois Theory for H-extensions and H-coextensions](#), [arXiv:0912.1785](#)) will be harmonized with this thesis, though it does not contain the results of the last chapter (pages [107–125](#)). The latter preprint has a long list of arxiv versions since my first attempt to build the Galois correspondence for extensions over rings turned out to contain a mistake. The results of Chapter 3 (pages [41–70](#)) are not included in either of the preprints, except for the completeness of lattices of generalised quotients and subalgebras of a Hopf algebra.

Chapter 1

Preliminaries

In this chapter we introduce the basic tools and notions we will use. We start with an introduction to *posets* (i.e. partially ordered sets), *lattices*, and *Galois connections*. Lattices are probably one of the most ubiquitous objects in mathematics. We recall the basics of the theory of *complete* and *algebraic lattices*. Complete lattices are the ones that allow for arbitrary infima and suprema, rather than just finite ones. We will use the existence theorem of Galois connections which requires completeness, and thus later on we will prove that the lattices that appear in the theory of Hopf algebras are complete. Algebraic lattices can be characterised as lattices of subalgebras of *universal algebras*. Also lattices of congruences of *universal algebras* are algebraic. For example all lattices of sub/quotient structures of all classical algebraic structures like: modules or groups, have algebraic lattices of sub/quotient structures. In one of the next chapters we are going to show that also some of the lattices which appear in the theory of Hopf algebras and their (co)actions are algebraic, usually under strong exactness properties of their underlying module (flat Mittag-Leffler modules, the property which will be studied in the next chapter). Every algebraic lattice is isomorphic to a lattice of subsets of a set which is closed under infinite intersections and directed sums (see Theorem 1.1.14 on page 10). We will use this theorem later, in order to show that some of the lattices are algebraic.

In section 1.2 (on page 11) we introduce the notion of a Galois connection between posets. The most important results of this section are: Proposition 1.2.2 (on page 12), where we show basic, but very useful, properties of Galois connections and the existence theorem of Galois correspondences between complete lattices (Theorem 1.2.6 on page 13). The main references for lattice theory and Galois connections are: Birkhoff and Frink [1948], Grätzer [1998], Davey and Priestley [2002] and Roman [2008].

In the final section 1.3 we collect basic definitions in the theory of Hopf algebras and their actions and coactions. You will find there definitions of a

coalgebra, a bialgebra and a Hopf algebra. We also introduce the theory of *co-modules* in Subsection 1.3 (on page 15). Then we pass to the main object of our study *comodule algebras* (Subsection 1.3 on page 20). Here we also recall the theory of crossed products. Their Galois theory we will study later on. Module algebras over a bialgebra or a Hopf algebras are recalled in Subsection 1.3 (on page 24). We close this section with a discussion (Subsection 1.3 on page 27) of finite Galois field extensions in terms of group Hopf algebras.

1.1 Posets and Lattices

Definition 1.1.1

A **partially ordered set**, **poset** for short, is a set P together with an order relation \preceq which is reflexive, transitive and antisymmetric.

A **down-set** D of P is a subset $D \subseteq P$ which satisfies: $d' \preceq d \in D \Rightarrow d' \in D$, dually an **upper-set** U of P is a subset $U \subseteq P$ with the property $U \ni u \preceq u' \Rightarrow u' \in U$.

Let (P, \preceq) and (Q, \leq) be two posets. Then a map $\phi : P \rightarrow Q$ is called **monotonic** if $p \preceq p'$ implies $\phi(p) \leq \phi(p')$ for $p, p' \in P$.

Let B be a subset of P . The smallest (greatest) element b of B , if it exists, is defined as the element of B such that for any $b' \in B$ $b \preceq b'$ (respectively, $b \succcurlyeq b'$). They are unique by the antisymmetry of an order relation. An element $p \in P$ is called an upper (lower) bound of B if for all $b \in B$ $p \succcurlyeq b$ (respectively, $p \preceq b$). We let $\sup B$ denote the least upper bound (supremum for short) of B and $\inf B$ the greatest lower bound (infimum) of B .

Definition 1.1.2

Let (P, \preceq) be a poset, then the **opposite poset** (P^{op}, \preceq^{op}) is defined by $P^{op} := P$ (as sets) and $p \preceq^{op} q \iff q \preceq p$ for $p, q \in P$. We will often write P^{op} to denote the opposite poset of a poset P .

Let (P, \preceq) and (Q, \leq) be two posets. A map $\phi : P \rightarrow Q$ is called **antimonotonic** if ϕ is a map of posets $\phi : (P^{op}, \preceq^{op}) \rightarrow (Q, \leq)$.

A **lattice** is a poset in which there exists the supremum and infimum of any subset with two elements or equivalently of any finite nonempty subset. A lattice can also be defined as an algebraic structure which has two binary operations: join (an abstract supremum of two elements) denoted by \vee and meet (an abstract infimum of two elements) denoted by \wedge which satisfy the following equalities:

$a \wedge a = a$	$a \vee a = a$	idempotent laws
$a \wedge b = b \wedge a$	$a \vee b = b \vee a$	commutative laws
$a \wedge (b \vee c) = (a \wedge b) \vee c$	$a \vee (b \wedge c) = (a \vee b) \wedge c$	associative laws
$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	absorption laws

The correspondence between lattice operations and the lattice order is made by: for a given order \preceq we define $a \wedge b := \inf\{a, b\}$ and $a \vee b := \sup\{a, b\}$, while for given lattice operations (\wedge, \vee) one defines an order by $a \preceq b \iff a = a \wedge b \iff b = a \vee b$. Let (L, \wedge_L, \vee_L) and (M, \wedge_M, \vee_M) be two lattices. Then a map $f : L \rightarrow M$ is called a **lattice homomorphism** if for any $k, l \in L$ we have $f(k \wedge_L l) = f(k) \wedge_M f(l)$ and $f(k \vee_L l) = f(k) \vee_M f(l)$. Note that a lattice homomorphism is necessarily an order preserving map, but the converse might not be true. A **lattice antihomomorphism** is a map $f : L \rightarrow M$ such that f is a homomorphism of lattices L^{op} and M , i.e. for any $k, l \in L$ we have $f(k \wedge_L l) = f(k) \vee_M f(l)$ and $f(k \vee_L l) = f(k) \wedge_M f(l)$.

If (L, \vee, \wedge) is a lattice then its dual is L, \wedge, \vee . Note that this definition is compatible with the definition of a dual poset.

We refer the reader to Grätzer [1998] for the theory of lattices.

Definition 1.1.3

A lattice (L, \vee, \wedge) is **complete** if for every non empty set $B \subseteq L$ there exists $\sup B$ and $\inf B$.

Remark 1.1.4 In a lattice L there exists arbitrary infima (suprema) if and only if there are arbitrary suprema (infima) of non empty subsets. Clearly, if a poset is closed under arbitrary infima (or suprema) then it is a complete lattice, since the following formulas hold

$$\begin{aligned} l \vee k &= \bigwedge \{l' \in L : l' \geq l, l' \geq k\} \quad \text{if } L \text{ is closed under infima,} \\ l \wedge k &= \bigvee \{l' \in L : l' \leq l, l' \leq k\} \quad \text{if } L \text{ is closed under suprema} \end{aligned}$$

for $l, k \in L$. We use the lattice notation for infima and suprema: $\bigwedge S$ denotes infimum of a subset $S \subseteq L$ and $\bigvee S$ its supremum.

Definition 1.1.5

1. A poset P which has finite infima is called a **lower semilattice**. A **filter** of a lower semilattice P is an upper-set which is closed under finite infima. The set of filters will be denoted by $\mathcal{F}(P)$.
2. A poset P which has finite suprema is called an **upper semilattice**. An **ideal** of an upper semilattice P is a down-set which is closed under finite suprema. The set of ideals will be denoted by $\mathcal{I}(P)$.

An upper semilattice is also called join-semilattice, and lower semilattice is called meet-semilattice. A **sublattice** of a lattice is a subset closed under meet and join. An **upper (lower) subsemilattice** of an upper (lower) semilattice is a subset which is closed under join (meet respectively).

Next we state a lemma which will be extensively used in the first part of this work.

Lemma 1.1.6

Let (M, \wedge_M, \vee_M) be a complete lattice and let there be two complete lattices which are upper sub-semilattices of M : (K, \wedge_K, \vee_M) and (L, \wedge_L, \vee_M) . Let K and L have the same smallest element. Then $(K \cap L, \wedge_{K \cap L}, \vee_{K \cap L})$, where

$$\begin{aligned} a \vee_{K \cap L} b &:= a \vee_M b \\ a \wedge_{K \cap L} b &:= \bigvee_M \{c \in K \cap L : c \leq a \wedge_M b\} \end{aligned}$$

is a complete lattice.

Proof: The join $\wedge_{K \cap L}$ is well defined, because K and L are upper sub-semilattices of the lattice M . Now let us prove that the meet is well defined as well. The lattices K and L have the same smallest element, so the set $\{c \in K \cap L : c \leq a \text{ and } c \leq b\}$, for any $a, b \in K \cap L$, is non-empty. The supremum $\bigvee_M \{c \in K \cap L : c \leq a \text{ and } c \leq b\}$ exists and belongs to $K \cap L$, because both K and L are complete. The axioms of lattice operations are trivially satisfied.

It remains to show that the lattice $(K \cap L, \wedge_{K \cap L}, \vee_{K \cap L})$ is complete. Let $B \subseteq K \cap L$. Then

$$\sup_{K \cap L} B = \sup_M B \quad \text{and} \quad \inf_{K \cap L} B = \sup_M \{x \in K \cap L : \forall b \in B \ x \leq b\}.$$

This infimum and this supremum exist, because M is complete and they belong to $K \cap L$ because K and L are complete upper sub-semilattices of M . \square

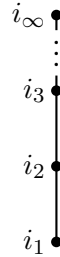
A subset $\{x \in L \mid a \leq x \leq b\}$ of the lattice L is called an **interval** and it will be denoted by $[a, b]$.

Definition 1.1.7

An element z of a lattice L is called **compact** if for any subset $S \subseteq L$ such that $z \leq \bigvee S$ there exists a finite subset S_f of S with the property $z \leq \bigvee S_f$.

Example 1.1.8 1. Let V be a k -vector space. Then a subspace W of V is a compact element of the lattice of subspaces $\text{Sub}_{\text{Vect}}(V)$ if and only if it is finite dimensional.

2. Let us consider the lattice $\mathbb{N} \cup \{\infty\}$:



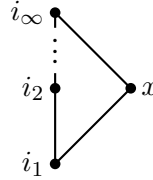
Then each element i_k for $k < \infty$ is compact and i_∞ is not compact though it is a supremum of compact elements.

Definition 1.1.9

A lattice is **algebraic** if it is complete and every element is a supremum of compact elements. A lattice is **dually algebraic** if its dual is algebraic.

Example 1.1.10

1. Let A be an \mathbf{R} -algebra, where \mathbf{R} is a commutative ring, then the lattices of ideals, left (right) ideals and subalgebras are algebraic.
2. The lattice from Example 1.1.8(ii) is an algebraic lattice.
3. The following complete lattice is not algebraic:



Then x is not a compact element, since $x \leq \bigvee_{k < \infty} i_k$ but for any $k_0 < \infty$ $\bigvee_{k \leq k_0} i_k = i_{k_0}$ is not comparable with x . The only compact elements are i_k for $k < \infty$. Furthermore, x is not a supremum of compact elements.

It is a well known theorem of Universal Algebra that lattices of subalgebras (see [Burris and Sankappanavar, 1981, Cor. 3.3]) and lattices of congruences (quotient structures, see [Burris and Sankappanavar, 1981, Thm. 5.5]) of any algebraic structure are algebraic. In particular, the lattices of sub-objects and quotient objects of classical algebraic structures like groups, semi-groups, rings, modules, etc. are algebraic. Furthermore, Birkhoff and Frink proved that any algebraic lattice can be represented as a lattice of subalgebras of a universal algebra (see Birkhoff and Frink [1948]).

Remark 1.1.11 Let $\text{Sub}_{\text{Vect}}(V)$ be the lattice of subspaces of a finite dimensional k -vector space V . By Example 1.1.8 every subspace of a finite dimensional vector space is a compact element of $\text{Sub}_{\text{Vect}}(V)$. Thus any sublattice of $\text{Sub}_{\text{Vect}}(V)$ is algebraic. It is dually algebraic as well, since we have a dual isomorphism:

$$\text{Sub}_{\text{Vect}}(V) \simeq \text{Sub}_{\text{Vect}}(V^*)$$

Thus a sublattice of $\text{Sub}_{\text{Vect}}(V)$ is anti-isomorphic to an algebraic sublattice of the lattice $\text{Sub}_{\text{Vect}}(V^*)$.

Definition 1.1.12

Let P be a poset. Then a non-empty set D of P is (upwards) **directed** if for every $a, b \in D$ there exists $c \in D$ such that $c \geq a$ and $c \geq b$.

Definition 1.1.13

Let X be a set and let $\mathcal{P}(X)$ denote the power set of X . Note that $\mathcal{P}(X)$ is a poset with the inclusion order of subsets of X . A subset $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a $\cap \vec{\cup}$ -**structure** if \mathcal{M} is closed under arbitrary intersections and unions of directed sets of its elements.

Directed unions of subsets we will denote by $\vec{\cup}$. The following theorem is an important characterisation of algebraic lattices.

Theorem 1.1.14 ([Roman, 2008, Thm. 7.5])

Let L be a lattice. Then L is algebraic if and only if it is isomorphic as a poset to a $\cap \vec{\cup}$ -structure.

The proof is after [Roman, 2008, Thm. 7.5]. We put it here since it gives an important point of view on the structure of algebraic lattices.

Proof: First let us observe that a $\cap \vec{\cup}$ -structure \mathcal{M} on a set X is an algebraic lattice. Since \mathcal{M} is closed under arbitrary intersections it is a complete lattice with meet \cap and join \cup . Now, let $S \subseteq X$ then we let $\langle S \rangle := \bigcap \{Y \in \mathcal{M} : S \subseteq Y\}$. We first show that $\langle S \rangle$ is a compact element of \mathcal{M} whenever S is finite. If $\langle S \rangle \subseteq \vec{\bigcup} \{M_i : M_i \in \mathcal{M}, i \in I\}$, for a finite set S , then for some $i \in I$ we have $S \subseteq M_i$ and thus $\langle S \rangle \subseteq M_i$. Conversely, let us assume that $K \in \mathcal{M}$ is a compact element. The family $\{\langle S \rangle : S \in \mathcal{M}, S \subseteq K, S \text{ is finite}\}$ is a directed set, since for $\langle S_i \rangle$ ($i = 1, 2$) we have $\langle S_i \rangle \subseteq \langle S_1 \cup S_2 \rangle \subseteq K$ for $i = 1, 2$. Now, we have

$$K \subseteq \vec{\bigcup} \{\langle S \rangle : S \subseteq K, S \text{ is finite}\}$$

And thus there exists a finite set S such that $K \subseteq \langle S \rangle$. The other inclusion is clear, since $S \subseteq K$. Thus we obtain $K = \langle S \rangle$ for some finite set S .

Now, consider the set $\mathcal{K}(L)$ of all compact elements of L . Then $\mathcal{K}(L)$ is an upper semilattice which inherits joins from the lattice L (i.e. supremum in L of compact elements is a compact element). Thus we can consider the set of ideals $\mathcal{I}(\mathcal{K}(L))$ of $\mathcal{K}(L)$, which is a $\cap \vec{\cup}$ -structure. Let $(J_i)_{i \in I}$ be a family of ideals then the intersection $\bigcap_{i \in I} J_i$ is an ideal: for if $l \in \bigcap_{i \in I} J_i$ then for every $i \in I$ $l \in J_i$ and thus if $l' \leq l$ then $l' \in J_i$ for every $i \in I$, and thus $l' \in \bigcap_{i \in I} J_i$. This shows that $\bigcap_{i \in I} J_i$ is a down set. If l and l' belong to $\bigcap_{i \in I} J_i$ then for every $i \in I$ we have $l, l' \in J_i$ and thus $l \vee l' \in J_i$ for all $i \in I$, hence $l \vee l' \in \bigcap_{i \in I} J_i$. This proves that the set of ideals is closed under all intersections. Now, let $(J_i)_{i \in I}$ be a directed system of ideals of $\mathcal{K}(L)$. We show that $\bigcup_{i \in I} J_i$ is an ideal. For this let $l \in \bigcup_{i \in I} J_i$ then there exists i such that $l \in J_i$. It follows that $\bigcup_{i \in I} J_i$ is a down set, since each J_i is. Let $l, l' \in \bigcup_{i \in I} J_i$. Then there exist $i, i' \in I$ such that $l \in J_i$ and $l' \in J_{i'}$. Furthermore, there exists $i'' \in I$ such that $J_i \cup J_{i'} \subseteq J_{i''}$, and thus $l, l' \in J_{i''}$. In consequence, $l \vee l' \in J_{i''} \subseteq \bigcup_{i \in I} J_i$. Hence $\bigcup_{i \in I} J_i$ is an ideal. This shows that the set of ideals of $\mathcal{K}(L)$ is indeed a $\cap \vec{\cup}$ -structure.

Note that $\emptyset \in \mathcal{K}(L)$ is the least element of $\mathcal{K}(L)$. We have a map: $L \ni a \mapsto \{k \in \mathcal{K}(L) : k \leq a\} \in \mathcal{I}(\mathcal{K}(L))$. It is clear that this map preserves the order. Let us take $a, a' \in L$, such that $\{k \in \mathcal{K}(L) : k \leq a\} = \{k \in \mathcal{K}(L) : k \leq a'\}$. Then we get

$$a = \sup\{k \in \mathcal{K}(L) : k \leq a\} = \sup\{k \in \mathcal{K}(L) : k \leq a'\} = a'$$

where the first and last equalities follow since L is an algebraic lattice. It remains to show that the considered map is an epimorphism. For this let $I \in \mathcal{I}(\mathcal{K}(L))$. Now let $a := \bigvee I$. Then $I \subseteq \{k \in \mathcal{K}(L) : k \leq a\}$. Now, if x belongs to the right hand side, then since $x \leq \bigvee I$, since x is compact, there exists $i \in I$ such that $x \subseteq i$, and thus $x \in I$ (since I is an ideal). Since the constructed map is a poset isomorphism it is also a lattice isomorphism. This ends the proof. \square

The following remark was proven at the beginning of the previous theorem.

Remark 1.1.15 *Let us stress that the compact elements of a $\cap\vec{\cup}$ -structure X are exactly the elements of the form $\langle S \rangle$ where $S \subseteq X$ is a finite subset.*

1.2 Galois connections

We start with a definition:

Definition 1.2.1 (Galois connection)

Let (P, \preceq) and (Q, \leq) be two partially ordered sets. Antimonotonic morphisms of posets $\phi : P \rightarrow Q$ and $\psi : Q \rightarrow P$ establish a **Galois connection** if

$$\forall_{p \in P, q \in Q} p \preceq \psi \circ \phi(p) \text{ and } q \leq \phi \circ \psi(q) \quad (1.1)$$

We refer to this property as the **Galois property**. An element of P (or Q) will be called **closed** if it is fixed by $\psi \circ \phi$ ($\phi \circ \psi$ respectively). The sets of closed elements of P and Q will be denoted by \overline{P} and \overline{Q} respectively. A standard notation for a Galois connection is

$$P \begin{smallmatrix} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{smallmatrix} Q$$

Another name which appear in the literature for this notion is **Galois correspondence**, which we use interchangeably.

Note that the Galois property (1.1) is equivalent to:

$$\phi(p) \geq q \iff p \preceq \psi(q) \quad (1.2)$$

for any $p \in P$ and $q \in Q$. Categorically speaking, Galois connections are the same as contravariant adjunctions between posets, which can be understood as categories in a straightforward manner. Note that in the definition of a Galois connection ϕ and ψ are not assumed to be lattice antihomomorphisms.

Proposition 1.2.2

Let (ϕ, ψ) be a Galois connection between posets (P, \preceq) and (Q, \leq) . Then:

1. $\bar{P} = \psi(Q)$ and $\bar{Q} = \phi(P)$,
2. The restrictions $\phi|_{\bar{P}}$ and $\psi|_{\bar{Q}}$ are inverse bijections of \bar{P} and \bar{Q} and \bar{P} and \bar{Q} are the largest subsets such that ϕ and ψ restricts to inverse bijections.
3. The map ψ is unique in the sense that if (ϕ, ψ) and (ϕ, ψ') form Galois connections then necessarily $\psi = \psi'$. In a similar way ϕ is unique.
4. The map ϕ is mono (onto) if and only if the map ψ is onto (mono).
5. If one of the two maps ϕ, ψ is an isomorphism then the second is its inverse.
6. The map ϕ is an injection if and only if $P = \bar{P}$.

Proof:

1. It is clear that $\bar{P} \subseteq \psi(Q)$, now let $p = \psi(q)$ for $q \in Q$. Then $\psi\phi(p) = \psi\phi\psi(q) \preceq \psi(q) = p$, since $\phi\psi(q) \geq q$, furthermore $\psi\phi(p) = \psi\phi\psi(q) \succeq \psi(q) = p$ by the Galois property (1.1) when applied to $\psi(q)$. The other equality $\bar{Q} = \phi(P)$ follows in the same way.
2. We showed that $\psi\phi\psi(q) = \psi(q)$; a similar argument shows that the following equality holds $\phi\psi\phi(p) = \phi(p)$. This together with (i) proves (ii).
3. Let both (ϕ, ψ) and $(\phi, \tilde{\psi})$ be Galois connections. We have $\psi \geq \psi\phi\tilde{\psi}$, since $id_Q \leq \phi\tilde{\psi}$ and $\psi\phi\tilde{\psi} \geq \tilde{\psi}$ since $\psi\phi \succeq id_P$. Thus $\psi \geq \tilde{\psi}$. Changing the roles of ψ and $\tilde{\psi}$ we will get $\tilde{\psi} \geq \psi$ and thus indeed $\psi = \tilde{\psi}$.
- (iv,v) Let us assume that ϕ is a monomorphism. Then $\psi\phi = id_P$ since we have $\phi\psi\phi = \phi$. Thus ψ is a split epimorphism. If ϕ is an epimorphism, then it follows that $\phi\psi = id_Q$ and thus ψ is a split monomorphism. This proves both (iv) and (v).
- (vi) The last assertion is a consequence of (iv).

□

Lemma 1.2.3

Let (ϕ, ψ) be a Galois connection between two posets (P, \preceq) and (Q, \leq) . Then ϕ and ψ reflect all existing suprema into infima.

Proof: Let $p_i \in P$ be such that $\bigvee p_i$ exists. Then $\phi(\bigvee p_i)$ is a lower bound of $\{\phi(p_i) : i \in I\}$. Furthermore, if b is a lower bound of $\{\phi(p_i) : i \in I\}$, i.e. $\phi(p_i) \geq b$ for all $i \in I$, then by (1.2) $p_i \preceq \psi(b)$ and thus $\bigvee p_i \preceq \psi(b)$. It follows that $\phi(\bigvee p_i) \geq b$, and so $\phi(\bigvee p_i) = \bigvee \phi(p_i)$. □

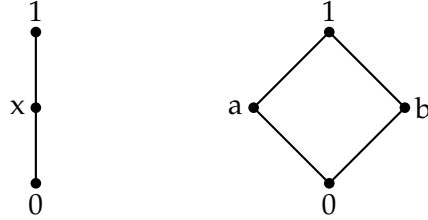
Corollary 1.2.4

Let (ϕ, ψ) be a Galois connection between two complete lattices (P, \succcurlyeq) and (Q, \geq) . Then the posets of closed elements are also complete.

Proof: It is enough to show that \overline{P} is closed under arbitrary infima. Let $S \subseteq \overline{P}$, then $S = \psi(W)$ for some $W \subseteq Q$. Now we have: $\inf_P S = \inf_P \psi(W) = \psi(\sup_Q W) \in \overline{P}$. \square

Below we construct an example of a Galois connection in which infima are not preserved, even if one of the maps is an antihomomorphism of lattices.

Example 1.2.5 Let us consider the following two lattices:



Then the maps $\phi(x) = a$, $\phi(1) = 0$, $\phi(0) = 1$, and $\psi(a) = x$, $\psi(b) = 0$, $\psi(1) = 0$, $\psi(0) = 1$ define a Galois correspondence. The closed elements are $\{1, x, 0\}$ and $\{1, a, 0\}$. Furthermore ϕ is an antihomomorphism of lattices (reflects suprema (infima) into infima (suprema)), but $\psi(a \wedge b) = \psi(0) = 1$, while $\psi(a) \vee \psi(b) = x \vee 0 = x$.

Theorem 1.2.6

Let P and Q be two posets. Let $\phi : P \rightarrow Q$ be an anti-monotonic map of posets. If P is complete then there exists a Galois connection (ϕ, ψ) if and only if ϕ reflects all suprema into infima.

Proof: After Lemma 1.2.3, it remains to prove that if ϕ reflects infinite suprema and P is complete then ϕ has a right adjoint.

Let ϕ be an antimonotonic map which reflects infinite suprema. Then ψ can be defined by the formula:

$$\psi(q) = \bigvee \{p \in P : \phi(p) \geq q\} \quad \forall q \in Q. \quad (1.3)$$

One then easily checks that:

$$\phi(p) \geq q \iff p \preceq \bigvee \{\tilde{p} \in P : \phi(\tilde{p}) \geq q\} := \psi(q)$$

\square

1.3 Algebras, coalgebras and Hopf algebras

The unadorned tensor product \otimes will denote the tensor product of modules over a commutative base ring \mathbf{R} .

Algebras

Definition 1.3.1

An **algebra** A over a ring \mathbf{R} is an \mathbf{R} -module together with a \mathbf{R} -bilinear associative multiplication: $m_A : A \times A \rightarrow A$. We consider all algebras to be unital, that is there exists a unit element of A , denoted by 1_A , such that $1_A \cdot a = a = a \cdot 1_A$ for all $a \in A$, where \cdot is the usual notation for a multiplication m_A . Furthermore, the homomorphism $1 : \mathbf{R} \rightarrow A, r \mapsto r1_A$ factors through the center of A . The associativity and unitality conditions can be expressed diagrammatically by imposing that the following diagrams commute:

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{m_A \otimes id_A} & A \otimes A \\
 id_A \otimes m_A \downarrow & & \downarrow m_A \\
 A \otimes A & \xrightarrow{m_A} & A
 \end{array}
 \quad
 \begin{array}{ccccc}
 \mathbf{R} \otimes A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A \otimes \mathbf{R} \\
 1 \otimes id_A \searrow & & \uparrow m_A & & \swarrow id_A \otimes 1 \\
 & & A \otimes A & &
 \end{array}$$

where $m_A : A \otimes A \rightarrow A$ is the \mathbf{R} -linear map defined by the \mathbf{R} -bilinear multiplication $\cdot : A \times A \rightarrow A$.

A morphism of \mathbf{R} -algebras $f : (A, m_A) \rightarrow (B, m_B)$ is an \mathbf{R} -module homomorphism from $f : A \rightarrow B$ such that $f(a \cdot a') = f(a) \cdot f(a')$.

Let us note that indeed if the above diagrams commute then the image of $1 : \mathbf{R} \rightarrow A$ lies in the center of A , since:

$$\begin{aligned}
 (r1_A) \cdot a &= m_A(r1_A \otimes a) = m_A(1_A \otimes ra) \\
 &= m_A(ra \otimes 1_A) = m_A(a \otimes r1_A) = a \cdot (r1_A)
 \end{aligned}$$

For $a, b \in A$ we will often write ab instead of $a \cdot b$ or $m_A(a \otimes b)$.

Coalgebras

Definition 1.3.2

A **coalgebra** C over a ring \mathbf{R} , an \mathbf{R} -coalgebra for short, is an \mathbf{R} -module, together with two maps: comultiplication $\Delta : C \rightarrow C \otimes C$ and counit $\epsilon : C \rightarrow \mathbf{R}$ such that the following diagrams commute:

$$\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\Delta \downarrow & & \downarrow \Delta \otimes id_C \\
C \otimes C & \xrightarrow{id_C \otimes \Delta} & C \otimes C \otimes C
\end{array}
\quad
\begin{array}{ccccc}
\mathbf{R} \otimes C & \xleftarrow{\cong} & C & \xrightarrow{\cong} & C \otimes \mathbf{R} \\
\epsilon \otimes id_C \swarrow & & \downarrow \Delta & & \searrow id_C \otimes \epsilon \\
& & C \otimes C & &
\end{array}
\quad (1.4)$$

Where $C \otimes \mathbf{R} \cong C \cong \mathbf{R} \otimes C$ are the canonical isomorphisms. We use the following version of the sumless Sweedler notation for the coproduct: $\Delta(c) = c_{(0)} \otimes c_{(1)}$; $\Delta(c)$ is an element of the tensor product $C \otimes C$. Thus it is a sum of simple tensors $\sum_{i=1}^n c'_i \otimes c''_i$. In the Sweedler notation this sum is written as $\sum_c c_{(1)} \otimes c_{(2)}$. We will also drop the summation sign and we will simply write $c_{(1)} \otimes c_{(2)}$. Note that particular simple tensors or even elements of C which appears in the sum are not unique.

A morphism $f : C \rightarrow D$ of coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ is a map of \mathbf{R} -modules such that $\Delta_D \circ f = f \otimes f \circ \Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.

Comodules

Definition 1.3.3

A (right) C -comodule M is an \mathbf{R} -module, together with a coaction defined by a map: $\delta : M \rightarrow M \otimes C$ such that:

$$\begin{array}{ccc}
M & \xrightarrow{\delta} & M \otimes C \\
\delta \downarrow & & \downarrow \delta \otimes id_C \\
M \otimes C & \xrightarrow{id_M \otimes \Delta} & M \otimes C \otimes C
\end{array}
\quad
\begin{array}{ccc}
M & \xrightarrow{\delta} & M \otimes C \\
\cong \searrow & & \downarrow id_M \otimes \epsilon \\
& & M \otimes \mathbf{R}
\end{array}$$

A morphism $f : M \rightarrow N$ of right C -comodules (M, δ_M) and (N, δ_N) is an \mathbf{R} -linear homomorphism such that $\delta_N \circ f = f \otimes id_C \circ \delta_M$.

In a similar way one defines left C -comodules and morphisms of them.

The category of right C -comodules with the above morphism will be denoted by Mod^C , the category of left C -comodules we will denote by ${}^C Mod$.

We will use the Sweedler's notation: for a right C -comodule M , $m \in M$, $\delta(m) = m_{(0)} \otimes m_{(1)}$ and for a left C -comodule N , $n \in N$, $\delta(n) = n_{(-1)} \otimes n_{(0)}$.

Then $C^* = \text{Hom}_{\mathbf{R}}(C, \mathbf{R})$ is a unital \mathbf{R} -algebra, with **convolution** as multiplication: $(f * g)(c) := f(c_{(1)})g(c_{(2)})$ for $f, g \in C^*$. Clearly, $\epsilon : C \rightarrow \mathbf{R}$ is the unit, since $(f * \epsilon)(c) = f(c_{(1)})\epsilon(c_{(2)}) = f(c_{(1)}\epsilon(c_{(2)})) = f(c)$ and similarly $\epsilon * f = f$. Associativity is a consequence of coassociativity of the comultiplication.

ation in C :

$$\begin{aligned}
((f * g) * h)(c) &= (f * g)(c_{(1)})h(c_{(2)}) \\
&= f(c_{(1)(1)})g(c_{(1)(2)})h(c_{(2)}) \\
&= f(c_{(1)})g(c_{(2)})h(c_{(3)}) \\
&= f(c_{(1)})g(c_{(2)(1)})h(c_{(2)(2)}) \\
&= f(c_{(1)})(g * h)(c_{(2)}) \\
&= (f * (g * h))(c)
\end{aligned}$$

Let M be a right C -comodule. Then M is a left $C^* = \text{Hom}_{\mathbf{R}}(C, \mathbf{R})$ module, with the action $c^* \cdot m := m_{(0)}c^*(m_{(1)})$ where $m \in M$ and $c^* \in C^*$. Associativity of this action follows from coassociativity of the C -coaction on M : $\delta : M \rightarrow M \otimes C$. Since the use of the dual space C^* we will assume that C is a projective \mathbf{R} -module.

Let M be a left C^* -module and let $\eta_M : C^* \otimes M \rightarrow M$, $\eta_M(c^* \otimes m) = c^*m$ for $c^* \in C^*$ and $m \in M$, be its module structure map. We also define

$$\rho_M : M \rightarrow \text{Hom}_{\mathbf{R}}(C^*, M), \rho_M(m)(c^*) := c^*m.$$

Let $ev : C \rightarrow C^{**}$ be the map $ev(c)(c^*) := c^*(c)$, and

$$f_M : M \otimes C^{**} \rightarrow \text{Hom}_{\mathbf{R}}(C^*, M), f_M(m \otimes c^{**})(c^*) = c^{**}(c^*)m$$

where $m \in M$, $c^* \in C^*$ and $c^{**} \in C^{**}$. Finally we define:

$$\mu_M : M \otimes C \rightarrow \text{Hom}_{\mathbf{R}}(C^*, M), \mu_M(m \otimes c)(c^*) = c^*(c)m$$

for $m \in M$, $c \in C$ and $c^* \in C^*$. It follows that μ_M is an injective map. This follows from projectivity of C : since we can choose a basis $\{d_i\}$ of C and a dual basis $\{d_i^*\}$ of C^* such that for every $c \in C$ $\sum_i d_i^*(c)d_i = c$. Now let $\mu_M(\sum_{\alpha} m_{\alpha} \otimes c_{\alpha}) = 0$, hence for every d_j^* we have $0 = \mu_M(\sum_{\alpha} m_{\alpha} \otimes c_{\alpha})(d_j^*) = d_j^*(\sum_{\alpha} c_{\alpha})m_{\alpha}$. Thus:

$$0 = \sum_{\alpha, j} d_j^*(c_{\alpha})m_{\alpha} \otimes d_j = \sum_{\alpha} m_{\alpha} \otimes c_{\alpha}$$

Definition 1.3.4

Let M be a left C^* -module with action $\eta_M : C^* \otimes M \rightarrow M$. We call it **rational** if the following inclusion holds:

$$\rho_M(M) \subseteq \mu_M(M \otimes C)$$

We let $\text{Rat}_{(C^* \text{ Mod})}$ denote the full subcategory of $C^* \text{ Mod}$ consisting of all rational modules.

Remark 1.3.5

1. Let M be a left C^* -module then it is rational if and only if there exist two finite families of $m_i \in M$ and $c_i \in C$ ($i = 1, \dots, n$) such that $c^*m = \sum_i m_i c^*(c_i)$ for every $m \in M$ and any $c \in C$.
2. Furthermore, if M is a C^* -rational left module and $\{m_i\}_{i=1, \dots, n}$, $\{c_i\}_{i=1, \dots, n}$ and $\{m'_i\}_{i=1, \dots, n}$, $\{c'_i\}_{i=1, \dots, n}$ are two such families then $\sum_{i=1}^n m_i \otimes c_i = \sum_{i=1}^n m'_i \otimes c'_i$ in $M \otimes C$, since $\mu_M(\sum_{i=1}^n m_i \otimes c_i) = \mu_M(\sum_{i=1}^n m'_i \otimes c'_i)$ and μ_M is injective.

By point (i) every C^* -module which comes from a C -comodule M is rational. The two families $\{m_i\}_{i=1, \dots, n}$ and $\{c_i\}_{i=1, \dots, n}$ of elements of M and C respectively, we get by imposing $\delta(m) = \sum_{i=1}^n m_i \otimes c_i$, where δ is the structure map of the C -comodule M . It turns out that the above remark allows to construct a C -comodule structure on a rational C^* -module. This leads to the following result.

Theorem 1.3.6

Let C be a \mathbf{k} -coalgebra. Then the functor which sends $M \in \text{Mod}^C$ to the corresponding rational C^* -module M is an isomorphism of categories

$$\text{Mod}^C \cong \text{Rat}(C^* \text{Mod}).$$

Proof: See [Dăscălescu et al., 2001, Thm 2.2.5]. □

For a corresponding statement for C -comodules over a commutative ring see Theorem 3.1.32 on page 54.

Let us note that if M is a rational C^* -module then so is any submodule and quotient module of it. See [Dăscălescu et al., 2001, Prop. 2.2.6] for the proof of this statement.

Finally, we will need the cotensor product of two \mathbf{R} -comodules. Let us consider a right C -comodule (M, δ_M) and a left C -comodule (N, δ_N) . Then $M \square_C N$ is the equaliser (in the category of \mathbf{R} -modules) of the following diagram:

$$M \square_C N \rightarrow M \otimes N \begin{array}{c} \xrightarrow{\delta_M \otimes id_N} \\ \xrightarrow{id_M \otimes \delta_N} \end{array} M \otimes C \otimes N$$

Bialgebras and Hopf algebras**Definition 1.3.7**

A **bialgebra** B is a tuple $(B, m, u, \Delta, \epsilon)$ such that (B, m, u) is an associative \mathbf{R} -algebra with multiplication $m : B \otimes B \rightarrow B$ and unit $u : \mathbf{R} \rightarrow B$, (B, Δ, ϵ) is an \mathbf{R} -coalgebra with comultiplication Δ and counit ϵ . Furthermore, the following compatibility conditions has to be satisfied, i.e. the following diagrams commute:

$$\begin{array}{ccccc}
B \otimes B & \xrightarrow{\Delta \otimes \Delta} & B \otimes B \otimes B \otimes B & \xrightarrow{id_B \otimes \tau \otimes id_B} & B \otimes B \otimes B \otimes B \\
\downarrow m & & & & \downarrow m \otimes m \\
B & \xrightarrow{\Delta} & & & B \otimes B
\end{array}$$

where $\tau : B^{\otimes 2} \rightarrow B^{\otimes 2}$ is the flip: $\tau(a \otimes b) = b \otimes a$,

$$\begin{array}{ccc}
B & \xrightarrow{\Delta} & B \otimes B \\
\uparrow u & & \uparrow u \otimes u \\
\mathbf{R} & \xrightarrow{\cong} & \mathbf{R} \otimes \mathbf{R}
\end{array}
\quad
\begin{array}{ccc}
B \otimes B & \xrightarrow{m} & B \\
\downarrow \epsilon \otimes \epsilon & & \downarrow \epsilon \\
\mathbf{R} \otimes \mathbf{R} & \xrightarrow{\cong} & \mathbf{R}
\end{array}$$

and finally: $\epsilon \circ u = id_{\mathbf{R}}$.

The commutative diagrams show that Δ and ϵ are morphisms of \mathbf{R} -algebras or equivalently that m and u are morphisms of R -coalgebras B and $B \otimes B$.

Definition 1.3.8

A **Hopf algebra** H is a tuple $(H, m, u, \Delta, \epsilon, S)$ such that $(H, m, u, \Delta, \epsilon)$ is a bialgebra and $S : H \rightarrow H$ is the antipode for which the following diagram commutes:

$$\begin{array}{ccccc}
& & H & & \\
& \swarrow \Delta & \downarrow \epsilon & \searrow \Delta & \\
H \otimes H & & \mathbf{R} & & H \otimes H \\
\downarrow S \otimes id_H & & \downarrow u & & \downarrow id_H \otimes S \\
H \otimes H & & H & & H \otimes H \\
& \searrow m & & \swarrow m & \\
& & H & &
\end{array}$$

or in Sweedler notation: $S(h_{(0)})h_{(1)} = \epsilon(h)1_H = h_{(0)}S(h_{(1)})$.

Let us note that the axioms of a Hopf algebra are self dual. If $(H, m, u, \Delta, \epsilon, S)$ is a finite dimensional Hopf algebra then its dual $(H^*, \Delta^*, \epsilon^*, m^*, u^*, S^{-1})$ is a Hopf algebra. The antipode of H^* is inverse of S , and it is well defined since for any finite dimensional Hopf algebra S is a bijection.

Examples 1.3.9 (Hopf algebras)

1. Let G be a group and let \mathbf{k} be a field. Then the group algebra $\mathbf{k}[G]$, which is a vector spaces spanned by G with multiplication induced from

the group multiplication, is a Hopf algebra, where $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. Elements c of a coalgebra C which satisfy the above conditions, i.e. $\Delta(c) = c \otimes c$ and $\epsilon(c)$ are called group-like. The antipode S in $k[G]$ is given by $S(g) = g^{-1}$ for every $g \in G$.

2. Let G be a finite group, then $k[G]^*$ is a Hopf algebra with basis $\delta_g \in k[G]^*$, which is a dual basis to the basis of $k[G]$ given by elements of G . The multiplication is given by: $\delta_g \cdot \delta_h = \delta_{g,h} \delta_g$, where $\delta_{g,h}$ is the Kronecker symbol given by $\delta_{g,h} = \begin{cases} 1 & \text{iff } g=h \\ 0 & \text{otherwise} \end{cases}$. The unit of the algebra structure is $\sum_{g \in G} \delta_g$ (note that we can write the unit in this form since G is finite). The comultiplication is set by:

$$\Delta(\delta_g) = \sum_{\substack{(h,k) \in G \times G \\ hk=g}} \delta_h \otimes \delta_k$$

and $\epsilon(\delta_g) := \delta_g(1) = \delta_{g,1}$. The antipode S is given by $S(\delta_g) = \delta(g^{-1})$.

3. An affine group scheme (over a field k) as a representable functor from the category of commutative algebras over k cAlg_k to the category of groups Gr . It turns out that the category of affine group schemes is equivalent to the category of commutative k -Hopf algebras denoted by cHopf_k via¹:

$$\text{cHopf}_k \ni H \mapsto \left(\text{cAlg}_k \ni A \mapsto \text{Hom}_{\text{cAlg}_k}(H, A) \in \text{Gr} \right)$$

The group structure on $\text{Hom}_{\text{cAlg}_k}(H, A)$ is the convolution product: for $f, g \in \text{Hom}_{\text{cAlg}_k}(H, A)$ $(f * g)(h) := f(h_{(1)}) \cdot g(h_{(2)})$ (where \cdot denotes the multiplication in the algebra A) and $f^{-1}(h) = f(S(h))$. Here S denotes the antipode of H . Affine group schemes form a category with morphisms being natural transformations which consist of group homomorphisms for every commutative algebra. This category we denote by GrSch_k .

Let us note that the group scheme represented by a Hopf algebra H is commutative (i.e. has values in commutative groups) if and only if H is a cocommutative Hopf algebra. The category of commutative group schemes, which is a full subcategory of GrSch_k , we denote by cGrSch_k .

4. Let \mathfrak{g} be a Lie algebra over a field k . Then the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is defined as a quotient of the unital free algebra $F(\mathfrak{g})$ on the set of basis elements of \mathfrak{g} by the ideal generated by $x \cdot y - y \cdot x - [x, y]$ for $x, y \in \mathfrak{g}$ where \cdot denotes multiplication in $F(\mathfrak{g})$ and $[-, -]$ is the Lie bracket of the Lie algebra \mathfrak{g} , i.e.

$$\mathcal{U}(\mathfrak{g}) := F(\mathfrak{g}) / \langle x \cdot y - y \cdot x - [x, y] : x, y \in \mathfrak{g} \rangle$$

¹This just follows from the Yoneda lemma. We refer to [Demazure and Gabriel \[1970\]](#) for the theory of affine group schemes.

There is a standard Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$. Indeed, this rule defines a map $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, which uniquely determines a map $\Delta' : F(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ since it is an algebra map and \mathfrak{g} generates $F(\mathfrak{g})$. This map in turn uniquely determines $\Delta : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ since:

$$\Delta'(x \cdot y - y \cdot x) = (x \cdot y - y \cdot x) \otimes 1 + 1 \otimes (x \cdot y - y \cdot x)$$

and $\Delta'([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y]$ since $[x, y] \in \mathfrak{g}$. The antipode S is uniquely determined by the rule $S(x) = -x$ for every $x \in \mathfrak{g}$.

H -comodule algebras

Definition 1.3.10

Let H be a \mathbf{R} -Hopf algebra, and A a \mathbf{R} -algebra and a right (left) H -comodule, i.e. there is a counital and coassociative map $\delta : A \rightarrow A \otimes H$ for the right H -comodule version. The algebra A is called an **H -comodule algebra** if the structure map δ is an algebra homomorphism.

Coinvariants of the H -coaction (H -comodule) are defined by $A^{co H} := \{a \in A : \delta(a) = a \otimes 1_H\}$. They form a subalgebra. An extension of the type $A/A^{co H}$ we will call an **H -extension**. Coinvariants of left H -comodule algebras are defined similarly.

Let us note that the coinvariants fit into the following equaliser diagram:

$$A^{co H} \rightrightarrows A \xrightarrow[\text{id}_A \otimes 1_H]{\delta} A \otimes H.$$

Examples 1.3.11 (Comodule algebras)

1. Every Hopf algebra H (over a commutative ring \mathbf{R}) is an H -comodule algebra both left and right. The H -comodule structure is set by the comultiplication Δ of H . Furthermore, $H^{co H} = \mathbf{R}$. For this let us assume that $h \in H^{co H}$, then $\Delta(h) = h_{(1)} \otimes h_{(2)} = h \otimes 1_H$. Now computing $\epsilon \otimes \text{id}_H$ of both sides we get $h = \epsilon(h)1 \in \mathbf{R}$. The other inclusion is clear.
2. Let G be a group and A a G graded \mathbf{R} -algebra. That is $A = \bigoplus_{g \in G} A_g$, and furthermore $A_g \cdot A_h \subseteq A_{gh}$. For $e \in G$ the unit element of G , A_e is a subalgebra of A . Then A is an $\mathbf{R}[G]$ -comodule algebra via: $\delta : A \rightarrow A \otimes \mathbf{k}[G]$, $\delta(a) = a \otimes g$ for all $a \in A_g$ for $g \in G$. Clearly, we have $A^{co \mathbf{R}[G]} = A_e$ since $e \in \mathbf{k}[G]$ is the unit element of the Hopf algebra $\mathbf{k}[G]$.

It is easy to observe that every $\mathbf{R}[G]$ -comodule algebra is of this form. For this one defines $A_g := \{a \in A : \delta(a) = a \otimes g\}$. Since δ is a monomorphism we get that $A_g \cap A_h = \{0\}$ if $g \neq h$ (for $g, h \in G$). The inclusion $A_g \cdot A_h \subseteq A_{gh}$ holds since δ is an algebra homomorphism.

The equality $A = \bigoplus_{g \in G} A_g$ follows now from coassociativity of the coaction δ : let $a \in A$, then $\delta(a) = \sum_{g \in G} a_g \otimes g$ for some $a_g \in A$ and $g \in G$, where only finitely many $a_g \neq 0$. Then by coassociativity of δ we get $\sum_{g \in G} \delta(a_g) \otimes g = \sum_{g \in G} a_g \otimes g \otimes g$. Now since G form an \mathbf{R} -basis of $\mathbf{R}[G]$ we get $\delta(a_g) = a_g \otimes g$, i.e. $a_g \in A_g$ for all $g \in G$. In this way $A = \bigoplus_{g \in G} A_g$.

3. Let B be an \mathbf{R} -algebra and H a Hopf algebra. Let us assume that H **acts weakly** on B that is there is a morphism $H \otimes B \rightarrow B$, $h \otimes a \mapsto h \cdot a$ such that:

- (a) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,
- (b) $h \cdot 1 = \epsilon(h)1$,
- (c) $1 \cdot a = a$.

Note that B might not to be an H -module, however if this is the case a weak action is called an **action**. Furthermore, let $\sigma : H \otimes H \rightarrow B$ be an \mathbf{R} -linear, convolution invertible map². The **crossed product** $B \#_\sigma H$ is $B \otimes H$ as an \mathbf{R} -module, with multiplication:

$$a \#_\sigma h \cdot b \#_\sigma k := a (h_{(1)} \cdot b) \sigma(h_{(2)} \otimes k_{(1)}) \#_\sigma h_{(3)} k_{(2)}$$

for all $a, b \in B$ and $h, k \in H$, where $a \#_\sigma h$ denotes $a \otimes h$ as an element of $B \#_\sigma H$. If the cocycle σ is trivial, i.e. $\sigma(-, -) = \epsilon(-) \otimes \epsilon(-)1_B$, then the crossed product $B \#_\sigma H$ is called a **smash product** and it is denoted by $B \# H$.

Lemma 1.3.12

The crossed product $B \#_\sigma H$ (over a commutative ring \mathbf{R}) is an associative algebra with unit $1_B \# 1_H$ if and only if the following two conditions are satisfied:

- (i) $1_H \cdot a = a$, for all $a \in B$, and

$$(h_{(1)} \cdot (k_{(1)} \cdot a)) \sigma(h_{(2)}, k_{(2)}) = \sigma(h_{(1)}, k_{(1)}) ((h_{(2)} k_{(2)}) \cdot a)$$

for all $h, k \in H$ and $a \in B$. B is then called a **twisted H -module**.

- (ii) $\sigma(h, 1_H) = \sigma(1_H, h) = \epsilon(h)1_B$, for all $h \in H$, and

$$(h_{(1)} \cdot \sigma(k_{(1)}, l_{(1)})) \sigma(h_{(2)}, k_{(2)} l_{(2)}) = \sigma(h_{(1)}, k_{(1)}) \sigma(h_{(2)} k_{(2)}, l)$$

for all $h, k, l \in H$. σ is then called a **cocycle**.

²That is there exists a map $\nu : H \otimes H \rightarrow B$ such that for all $h \otimes k \in H \otimes H$ we have $\sigma(h_{(1)} \otimes k_{(1)}) \nu(h_{(2)} \otimes k_{(2)}) = \epsilon(h) \epsilon(k) 1_B = \nu(h_{(1)} \otimes k_{(1)}) \sigma(h_{(2)} \otimes k_{(2)})$

See [Doi and Takeuchi, 1986, Lem. 10] or Blattner et al. [1986]. In the rest of this work we assume that all crossed products are associative with identity $1_B \#_\sigma 1_H$, and thus that the conditions of the lemma are satisfied.

Now, $B \#_\sigma H$ becomes an H -comodule algebra by setting:

$$\delta(a \#_\sigma h) = a \#_\sigma h_{(1)} \otimes h_{(2)}, \text{ for } a \in B, h \in H.$$

Definition 1.3.13

An H -extension $A/A^{\text{co } H}$ is called **H -Hopf-Galois extension** (**H -Galois extension**) if the **canonical map** of right H -comodules and left A -modules:

$$\text{can} : A \otimes_{A^{\text{co } H}} A \rightarrow A \otimes H, \quad a \otimes b \mapsto ab_{(0)} \otimes b_{(1)} \quad (1.5)$$

is an isomorphism.

Examples 1.3.14 (Hopf-Galois extensions)

- Ex. 1.3.11 (i) Let H be a Hopf algebra, then it is an H -Galois extension of the base ring \mathbf{R} with $\text{can}^{-1}(h \otimes k) := hS(k_{(1)}) \otimes k_{(2)}$.
- Ex. 1.3.11 (ii) A G -graded \mathbf{k} -algebra A is a $\mathbf{k}[G]$ -Galois extension if and only if A is strongly graded, that is $A_g \cdot A_h = A_{gh}$. This was first discovered by Ulbrich [1982].
- Ex. 1.3.11 (iii) A crossed product $B \#_\sigma H$ over a ring \mathbf{R} is a Hopf-Galois extension as we show below.

Definition 1.3.15

- An H -extension A/B (over a ring \mathbf{R}) is called **cleft** if there exists a convolution invertible H -comodule map $\gamma : H \rightarrow A$.
- An H -extension A/B has the **normal basis property** if and only if A is isomorphic to $B \otimes H$ as a left B -module and right H -comodule.

Let us note that the normal basis property is equivalent to the classical notion if H is finite dimensional. We defer the explanation until introducing H -module algebras. We say that a R -algebra A is flat if it is flat R -module, that is the functor $A \otimes - : \text{Mod}_R \rightarrow \text{Mod}_R$ preserves exact sequences.

Theorem 1.3.16

Let A be an H -comodule algebra over \mathbf{R} such that $B = A^{\text{co } H}$ is a flat \mathbf{R} -algebra. Then the following are equivalent:

1. the extension $B \subseteq A$ is equivalent to a crossed product $B \subseteq B \#_\sigma H$ for some weak action of H on B and some invertible cocycle σ satisfying the twisted module condition;

2. $B \subseteq A$ is cleft;
3. $B \subseteq A$ is H -Galois with normal basis property.

This theorem generalises [Blattner and Montgomery, 1989, Thm 1.18] where it is proven for algebras over a field k . Let us note that the proof of Blattner and Montgomery [1989] also works in the more general context, however we show a different proof.

Proof: The equivalence between (ii) and (iii) was shown in Doi and Takeuchi [1986]. The implication (ii) \Rightarrow (i) follows from [Doi and Takeuchi, 1986, Thm 11] as shown in the proof of [Blattner and Montgomery, 1989, Thm 1.18]. Instead of proving (i) \Rightarrow (ii) as it is done by Blattner and Montgomery, we show that (i) \Rightarrow (iii).

First, if B is flat over \mathbf{R} we have $(B \#_{\sigma} H)^{co H} = B \#_{\sigma} 1_H \subseteq B \#_{\sigma} H$: the diagram

$$\mathbf{R} = H^{co H} \xrightarrow{\quad} H \xrightleftharpoons[id_H \otimes 1_H]{\Delta} H \otimes H$$

is an equaliser and B is flat over \mathbf{R} thus the fork

$$B = B \otimes \mathbf{R} \xrightarrow{\quad} B \#_{\sigma} H \xrightleftharpoons[id_{B \#_{\sigma} H} \otimes 1_H]{\delta} B \#_{\sigma} H \otimes H$$

is an equaliser as well. Hence, indeed, $(B \#_{\sigma} H)^{co H} = B \#_{\sigma} 1_H \cong B$. Now it easily follows that $B \#_{\sigma} H$ has the normal basis property, since $a \#_{\sigma} 1 \cdot b \#_{\sigma} h = (ab) \#_{\sigma} h$, for $a, b \in B$, $h \in H$, and $\delta = id_B \otimes \Delta : B \#_{\sigma} H \rightarrow B \#_{\sigma} H \otimes H$. To prove the Galois property let us observe that we have a commutative diagram:

$$\begin{array}{ccc} (B \#_{\sigma} H) \otimes_B (B \#_{\sigma} H) & \xrightarrow{can_{B \#_{\sigma} H}} & B \#_{\sigma} H \otimes H \\ \alpha \downarrow & \nearrow \beta & \\ B \otimes H \otimes H & & \end{array}$$

where $\alpha : (B \#_{\sigma} H) \otimes_B (B \#_{\sigma} H) \rightarrow B \#_{\sigma} H \otimes H$ is defined by: $\alpha(a \#_{\sigma} h \otimes_B b \#_{\sigma} k) = (a \#_{\sigma} h \cdot b \#_{\sigma} 1_h) \otimes k = a(h_{(1)} \cdot b) \#_{\sigma} h_{(2)} \otimes k$. It is an isomorphism with inverse $B \#_{\sigma} H \otimes H \ni a \otimes h \otimes k \mapsto a \#_{\sigma} h \otimes 1_A \#_{\sigma} k \in (B \#_{\sigma} H) \otimes_B (B \#_{\sigma} H)$. The map β is explicitly given by:

$$\begin{aligned} B \otimes H \otimes H \ni a \otimes h \otimes k &\xrightarrow{\beta} a \sigma(h_{(1)}, k_{(1)}) \otimes can_H(h_{(2)} \otimes k_{(2)}) \\ &= a \sigma(h_{(1)}, k_{(1)}) \otimes h_{(2)} k_{(2)} \otimes k_{(3)} \in B \#_{\sigma} H \otimes H \end{aligned}$$

It is invertible with inverse β^{-1} the composition:

$$\begin{aligned} B \#_{\sigma} H \otimes H \ni a \#_{\sigma} h \otimes k &\mapsto a \otimes \text{can}_H^{-1}(h \otimes k) \\ &= a \otimes hS(k_{(1)}) \otimes k_{(2)} \\ &\mapsto a\sigma^{-1}(h_{(1)}S(k_{(2)}), k_{(3)}) \otimes h_{(2)}S(k_{(1)}) \otimes k_{(4)} \\ &\in B \otimes H \otimes H \end{aligned}$$

where σ^{-1} is the convolution inverse of σ . It now follows that $\text{can}_{B \#_{\sigma} H}$ is invertible and thus $B \subseteq B \#_{\sigma} H$ is H -Galois with the normal basis property. \square

Corollary 1.3.17

Let A/B be an H -extension, such that both A and H are finite dimensional. Furthermore, let us assume that A has the normal basis property. Then A is a crossed product if and only if can_A is an epimorphism or monomorphism.

Proof: Since $A \cong B \otimes H$ as a left B -module and a right H -comodule we have $A \otimes_B A \cong A \otimes_B (B \otimes H) \cong A \otimes H$. Thus $\dim_k A \otimes_B A = \dim_k A \otimes H$. It follows that $\text{can}_A : A \otimes_B A \rightarrow A \otimes H$ is an isomorphism (and hence A is a crossed product) if and only if it is a monomorphism or an epimorphism. \square

H -module algebras

Another set of examples of H -comodule algebras can be built from H -module algebras.

Definition 1.3.18

Let A be an algebra and H a Hopf algebra. We say that A is an H -module algebra if H measures A , that is there is an action $H \otimes A \rightarrow A$ such that $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ which makes A an H -module, or equivalently the following diagram commutes:

$$\begin{array}{ccc} H \otimes A \otimes A & \xrightarrow{m} & H \otimes A \\ \downarrow & & \downarrow \\ A \otimes A & \xrightarrow{m} & A \end{array} \quad (1.6)$$

where $m : A \otimes A \rightarrow A$ is the multiplication of A and the vertical maps are the H -action. The **invariant** subalgebra A^H is defined as $A^H := \{a \in A : \forall h \in H h \cdot a = \epsilon(h)a\}$.

Let us recall the following theorem:

Proposition 1.3.19

Let H be a finite dimensional Hopf algebra (over a field \mathbf{k}), and let A be a \mathbf{k} -algebra. Then A is a right H -comodule algebra if and only if A is a left H^* -module algebra. Furthermore, in this case we have $A^{co H} = A^{H^*}$.

Proof: The proof can be found in [Dăscălescu et al., 2001, Thm 6.2.4]. We only sketch the constructions. Let A be an H -comodule algebra with the H -comodule structure map $\delta : A \rightarrow A \otimes H$. Then A is an H^* -module algebra by:

$$f \cdot a := id_A \otimes f \circ \delta(a) = a_{(0)} f(a_{(1)})$$

Conversely, if A is an H^* -module algebra, then we pick a dual basis of H^* and H : $\{e_i^*\}$ and $\{e_i\}$ respectively, where $i = 1, \dots, n = \dim H$. Then the H -comodule structure map is set by:

$$\delta : A \rightarrow A \otimes H, \quad \delta(a) = \sum_{i=1}^n (e_i^* \cdot a) \otimes e_i.$$

□

We are now ready to explain the connection between the non-classical and classical normal basis properties (see Definition 1.3.15).

Lemma 1.3.20

Let A be an H -comodule algebra over a field \mathbf{k} , with H a finite dimensional Hopf algebra. Let us consider A as an H^* -module algebra. Then A has the normal basis property if and only if there exists $a \in A$ and a basis $\{f_i\}_{i=1, \dots, n}$ of H^* such that $\{f_i \cdot a\}_{i=1, \dots, n}$ is a basis of A as a left H -module.

Proof: A sketch of the proof can be found in [Montgomery, 2009, Lemma 3.5].

□

Examples 1.3.21 (H -module algebras)

1. Let G be a group acting by automorphisms on an \mathbf{R} -algebra A . Then A is an $\mathbf{R}[G]$ -module algebra via the induced action: $\mathbf{R}[G] \otimes A \rightarrow A$ from the G -action. Furthermore, $\mathbf{R}[G]$ -module structures on A are in one to one correspondence with actions of G on A via the automorphism group of the algebra A . In this case $A^{k[G]}$ is the subalgebra of elements of A which are fixed by all elements of G (sum of all one element orbits of the G -action).
2. If in the previous example we assume that G is a finite group and we assume that $\mathbf{R} = \mathbf{k}$ is a field, we get that A is a $\mathbf{k}[G]^*$ -comodule algebra (by Proposition 1.3.19) and $\mathbf{k}[G]^*$ -comodule algebra structures on A are in one to one correspondence with G -actions on A via the automorphism group of A . It also follows that $A^{co \mathbf{k}[G]^*}$ is the algebra of fixed elements of the corresponding G -action.

3. This is a dual case of Example 1.3.11(ii). Let, as before, A be a G graded \mathbf{k} -algebra, where G is a finite group. Let $H = \mathbf{k}[G]^*$, and let $\{e_g : g \in G\}$ be the dual basis of $\mathbf{k}[G]^*$ to the basis $\{g : g \in G\}$ of $\mathbf{k}[G]$. The comultiplication of $\mathbf{k}[G]^*$ can be described by the following formula:

$$\Delta(e_g) = \sum_{h \in G} e_{gh^{-1}} \otimes e_h$$

the counit is set by $\epsilon(e_g) = 0$ if $g \neq e$, where $e \in G$ is the unit element of G and $\epsilon(e) = 1$. The antipode is set by $S(e_g) = e_{g^{-1}}$. For $a \in A$ we have $a = \sum_{g \in G} a_g$ where $a_g \in A_g$ for all $g \in G$. The $\mathbf{k}[G]^*$ -module algebra structure on $A = \bigoplus_{g \in G} A_g$ is set by $e_g \cdot a = a_g$, where a_g is the A_g component of a . Indeed, the H -module algebra condition is satisfied, since

$$e_g \cdot (ab) = (ab)_g = \sum_{h \in G} a_{gh^{-1}} b_h = \sum_{h \in G} (e_{gh^{-1}} \cdot a)(e_h \cdot b)$$

The subalgebra of invariants is A_e (where $e \in G$ is the unit element of the group).

4. Let \mathfrak{g} be a \mathbf{k} -Lie algebra and let A be a \mathbf{k} -algebra such that \mathfrak{g} acts on A by derivations, i.e. there exists a homomorphism $\alpha : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(A)$. Here $\text{Der}_{\mathbf{k}}(A)$ denotes the set of \mathbf{k} -linear endomorphisms of A , such that $f(ab) = f(a)b + af(b)$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping (Hopf) algebra of \mathfrak{g} . Since $\mathcal{U}(\mathfrak{g})$ is generated (as a vector space) by monomials $g_1 \dots g_n$, $g_i \in \mathfrak{g}$, we put

$$(g_1 \dots g_n) \cdot a = \alpha(g_1)(\dots \alpha(g_n)(a) \dots) \quad (1.7)$$

In this way A becomes a $\mathcal{U}(\mathfrak{g})$ -module algebra. Furthermore, this sets up a one to one correspondence between $\mathcal{U}(\mathfrak{g})$ -comodule algebra structures on A and \mathfrak{g} actions through derivations on A . If $\mathcal{U}(\mathfrak{g}) \otimes A \rightarrow A$ is a $\mathcal{U}(\mathfrak{g})$ -module algebra structure then for each $g \in \mathfrak{g}$ we have: $g \cdot (ab) = (g_{(1)} \cdot a)(g_{(2)} \cdot b) = (g \cdot a)b + a(g \cdot b)$ since $\Delta(g) = g \otimes 1 + 1 \otimes g$ in $\mathcal{U}(\mathfrak{g})$. Thus a $\mathcal{U}(\mathfrak{g})$ -module algebra structure restricts to an action of \mathfrak{g} through derivations, which is the one we started with. Conversely, if we have an $\mathcal{U}(\mathfrak{g})$ -module algebra structure, then it must satisfy the formula 1.7, since it is associative (it is a $\mathcal{U}(\mathfrak{g})$ -module). Thus a $\mathcal{U}(\mathfrak{g})$ -module algebra structure is uniquely determined by the underlying \mathfrak{g} -action.

5. There is an adjoint action of H on itself, which makes H an H -module algebra:

$$h \cdot k := \text{ad}(h)(k) := h_{(1)}kS(h_{(2)}).$$

Indeed:

$$\begin{aligned} \text{ad}(h)(kl) &:= h_{(1)}klS(h_{(2)}) = h_{(1)}kS(h_{(2)})h_{(3)}lS(h_{(4)}) \\ &= \text{ad}(h_{(1)})(k)\text{ad}(h_{(2)})(l) \end{aligned}$$

In the case $H = \mathbf{k}[G]$ it is given by $h \cdot k := hkh^{-1}$ for $h, k \in G$, and in the case $H = \mathcal{U}(\mathfrak{g})$ it is determined by the adjoint action: $\text{ad}(g)(h) := gh - hg$ for $g \in \mathfrak{g}, h \in H$. We have $H^H = Z(H)$ the center of H . Indeed, for $h \in H^H$ and $k \in H$ we have

$$kh = k_{(1)}\epsilon(k_{(2)})h = k_{(1)}hS(k_{(2)})k_{(3)} = \epsilon(k_{(1)})hk_{(2)} = hk.$$

The other inclusion easily follows from the antipode axiom (see Definition 1.3.8).

Finite Galois field extensions

In this subsection we introduce Galois field extensions and show how they can be understood using the Hopf–Galois approach.

Definition 1.3.22

Let \mathbb{E}/\mathbb{F} be an algebraic field extension. It is called **Galois** if there exists a subgroup $G < \text{Aut}(\mathbb{E})$ such that $\mathbb{F} = \mathbb{E}^G := \{e \in \mathbb{E} : \forall g \in G, g(e) = e\}$.

If \mathbb{E}/\mathbb{F} is a Galois extension then it turns out that the group $G = \text{Aut}(\mathbb{E}/\mathbb{F})$, i.e. the group of automorphisms of \mathbb{E} which preserve the subfield \mathbb{F} . Let \mathbb{E}/\mathbb{F} be a field extension. Let us denote by $\text{Gal}(\mathbb{E}/\mathbb{F})$ the subgroup of $\text{Aut}(\mathbb{E})$ of all automorphisms ϕ such that for all $f \in \mathbb{F}$ $\phi(f) = f$. We let $[\mathbb{E} : \mathbb{F}] := \dim_{\mathbb{F}} \mathbb{E}$ denote the **degree** of the field extension \mathbb{E}/\mathbb{F} . An extension is called **finite** if the degree is finite. It turns out that an algebraic extension is Galois if and only if $\mathbb{E}^{\text{Gal}(\mathbb{E}/\mathbb{F})} = \mathbb{F}$ and as a consequence \mathbb{E}/\mathbb{F} is a finite Galois extension if and only if $[\mathbb{E} : \mathbb{F}] = |\text{Gal}(\mathbb{E}/\mathbb{F})|$. Furthermore, let us note that Galois extensions have the classical basis property (see [Jacobson, 1985, chapter 4.14]), i.e. there exists $e \in \mathbb{E}$ such that $\{g(e) : g \in \text{Gal}(\mathbb{E}/\mathbb{F})\}$ is a basis over \mathbb{F} .

Now let us assume that \mathbb{E}/\mathbb{F} is a finite field extension, and thus $\text{Gal}(\mathbb{E}/\mathbb{F})$ is a finite group. We already know that the $\text{Gal}(\mathbb{E}/\mathbb{F})$ -action on \mathbb{E} induces a $\mathbb{F}[\text{Gal}(\mathbb{E}/\mathbb{F})]$ -module algebra structure on \mathbb{E} . Furthermore, let $\mathbf{k} \subseteq \mathbb{F}$ be a finite field extension. Then \mathbb{E} possesses a $\mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]$ -module algebra structure.

Proposition 1.3.23

Let \mathbb{E}/\mathbb{F} be a finite field extension and $\mathbf{k} \subseteq \mathbb{F}$ be as above. Then the extension \mathbb{E}/\mathbb{F} is Galois if and only if it is a $\mathbf{k}[G]^*$ -Hopf–Galois extension for some group G , i.e. $\mathbb{F} = \mathbb{E}^{\text{co } \mathbf{k}[G]^*}$ and the canonical map

$$\text{can}_{\mathbb{E}} : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \longrightarrow \mathbb{E} \otimes \mathbf{k}[G]^*$$

is bijective. Furthermore, if this is the case, then $G = \text{Gal}(\mathbb{E}/\mathbb{F})$.

Though the proof is well known (see for example Montgomery [2009]), for the sake of completeness we put it below.

Proof: First assume that \mathbb{E}/\mathbb{F} is a Galois extension (Definition 1.3.22). Let $n = |\text{Gal}(\mathbb{E}/\mathbb{F})|$, let $\text{Gal}(\mathbb{E}/\mathbb{F}) = \{g_1, \dots, g_n\}$ and let $\{\phi_i\}_{i=1, \dots, n}$ be the dual

basis of $\mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]$. Since \mathbb{E}/\mathbb{F} is Galois it has an \mathbb{F} -basis $\{f_i\}_{i=1,\dots,n}$ of n elements. The $\text{Gal}(\mathbb{E}/\mathbb{F})$ -action on \mathbb{E} determines a coaction as shown in Proposition 1.3.19, that is $\delta : \mathbb{E} \rightarrow \mathbb{E} \otimes \mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*$, $\delta(e) = \sum_{i=1}^n g_i(e) \otimes \phi_i$. Furthermore, $\mathbb{F} = \mathbb{E}^{\text{Gal}(\mathbb{E}/\mathbb{F})} = \mathbb{E}^{\text{co } \mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*}$. The canonical map is given by $\text{can}_{\mathbb{E}}(e \otimes_{\mathbb{F}} e') = \sum_{i=1}^n e g_i(e') \otimes \phi_i$. Thus if $\sum_{i=1}^n e_i \otimes f_i \in \ker \text{can}_{\mathbb{E}}$, then $\sum_{i=1}^n e_i g_j(f_i) = 0$ for all $j = 1, \dots, n$, since $\{\phi_j\}_{j=1,\dots,n}$ are all independent. Now let us consider the $n \times n$ -matrix \mathbb{A} with entries $\mathbb{A}_{ij} := g_j(e_i)$. The Dedekind lemma on independence of automorphisms implies that its columns are independent, and thus it is invertible. It follows that $e_i = 0$ for all $i = 1, \dots, n$. Thus $\text{can}_{\mathbb{E}}$ is injective. Now, $\dim_{\mathbb{F}}(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) = [\mathbb{E} : \mathbb{F}]^2$, hence $\dim_{\mathbf{k}}(\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) = [\mathbb{E} : \mathbb{F}]^2 [\mathbb{F} : \mathbf{k}]$ and $\dim_{\mathbf{k}}(\mathbb{E} \otimes \mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]) = [\mathbb{E} : \mathbf{k}] |\text{Gal}(\mathbb{E}/\mathbb{F})| = [\mathbb{E} : \mathbb{F}]^2 [\mathbb{F} : \mathbf{k}]$. The canonical map $\text{can}_{\mathbb{E}}$ is an isomorphism since both $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ and $\mathbb{E} \otimes \mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]$ are of the same finite dimension.

Now, let us assume that \mathbb{E}/\mathbb{F} is $\mathbf{k}[G]$ -Galois. Then by Proposition 1.3.19 there exists a $\mathbf{k}[G]$ -module algebra structure on \mathbb{E} , with $\mathbb{E}^{\mathbf{k}[G]} = \mathbb{E}^{\text{co } \mathbf{k}[G]^*} = \mathbb{F}$. It is uniquely determined by a G -action on \mathbb{E} and moreover $\mathbb{E}^G = \mathbb{E}^{\mathbf{k}[G]} = \mathbb{F}$. Thus \mathbb{E}/\mathbb{F} is a Galois extension. Since $\text{can}_{\mathbb{E}}$ is an isomorphism comparing the \mathbf{k} -dimensions of $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ and $\mathbb{E} \otimes \mathbf{k}[G]$ we get $[\mathbb{E} : \mathbb{F}] = |G|$. By [Kreimer and Takeuchi, 1981, Thm 1.7] we get that

$$\mathbb{E} \otimes \mathbf{k}[G] \rightarrow \text{End}_{\mathbb{F}}(\mathbb{E}), \quad e \otimes g \mapsto (\mathbb{E} \ni e' \mapsto e g(e') \in \mathbb{E})$$

is an isomorphism. This implies that the map $G \ni g \mapsto (e \mapsto g(e)) \in \text{Aut}(\mathbb{E})$ is a monomorphism. In consequence we can write $G \leq \text{Aut}(\mathbb{E})$. Furthermore, $G \leq \text{Gal}(\mathbb{E}/\mathbb{F})$, since $\mathbb{E}^G = \mathbb{F}$. Now, since $|G| = [\mathbb{E} : \mathbb{F}] = |\text{Gal}(\mathbb{E}/\mathbb{F})|$ (since \mathbb{E}/\mathbb{F} is a Galois extension) we get that $G = \text{Gal}(\mathbb{E}/\mathbb{F})$. \square

Corollary 1.3.24

Let \mathbb{E}/\mathbb{F} be a finite Galois extension of fields. Then \mathbb{E} is a crossed product, i.e. there exists a cocycle σ and a weak action of $\mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*$ on \mathbb{F} such that

$$\mathbb{E} \cong \mathbb{F} \#_{\sigma} \mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*.$$

Proof: Finite Galois extensions are $\mathbf{k}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*$ -Hopf-Galois and have a normal basis, thus the result follows from Theorem 1.3.16. \square

Chapter 2

Modules with the intersection property

In this chapter we investigate some properties of the functor $M \otimes -$ for an \mathbf{R} -module M . The result obtained will play a crucial role in further studies. We first recall a result that for a flat \mathbf{R} -module M the functor $M \otimes -$ preserves all finite intersections. Then we consider a new property, which we call the *intersection property* (Definition 2.1.26 on page 31). A module has this property if the tensor functor preserves all intersections, not just finite ones. This property will be used later to show that the lattice of subcomodules of a C -comodule is algebraic if the coalgebra C is a flat Mittag-Leffler module. Moreover, this property will play a vital role in the construction of a Galois correspondence for comodule algebras over a ring. We show that direct sums of modules which have the intersection property also have it, thus free modules have it and we show that direct summands of modules with this property also share it. Hence projective modules have the intersection property. In Proposition 2.1.31 (on page 34) we show that flat Mittag-Leffler modules also have the intersection property. The Mittag-Leffler condition was first studied in [Raynaud and Gruson, 1971]. The authors have shown that flat Mittag-Leffler modules preserve filtered limits of monomorphisms (Proposition 2.1.30 on page 33). Since intersection is this kind of limit flat Mittag-Leffler modules have the intersection property. We give here an independent proof of this fact. Based on the results of Raynaud and Gruson we note that flat modules have this property if and only if they satisfy the Mittag-Leffler condition (Corollary 2.1.36 on page 35). Herbera and Trlifaj [2009] showed that κ -projective modules (see Definition 2.1.33 on page 35) are flat Mittag-Leffler and hence they possess the intersection property. We also prove that locally projective modules (Definition 2.1.39 on page 37) have this property. Then we show that the intersection property is stable under pure submodules (Proposition 2.1.38 on page 36). The very last Theorem 2.1.40 (on page 38) shows a condition that must be fulfilled in order for every flat R -module to have the intersection property. Though we rather tend to think

that this theorem might be useful the other way around. Knowing a ring for which all the flat modules have the intersection property we gain another tool (a class of exact sequences) to study its modules. For example every flat right R -module is projective (and hence has the intersection property) if (and only if) the ring is right perfect (by a theorem of Bass).

Proposition 2.1.25 ([Brzeziński and Wisbauer, 2003, 40.16(2)])

Let $M', M'' \subseteq M$ be \mathbf{R} -submodules and let K be a flat \mathbf{R} -module. Then

$$(M' \otimes K) \cap (M'' \otimes K) = (M' \cap M'') \otimes K$$

Since the proof in Brzeziński and Wisbauer [2003] is omitted we put it here.

Proof: Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (M' \cap M'') \otimes K & \xrightarrow{i_1} & M' \otimes K & \xrightarrow{p' \otimes id_K} & M'/(M' \cap M'') \otimes K \longrightarrow 0 \\
 & & \downarrow i_2 & & \downarrow i' \otimes id_K & & \downarrow \\
 0 & \longrightarrow & M'' \otimes K & \xrightarrow{i'' \otimes id_K} & M \otimes K & \xrightarrow{p'' \otimes id_K} & M/M'' \otimes K \longrightarrow 0
 \end{array}$$

where both rows are exact and every vertical map is a monomorphism (monomorphisms in the category of modules are just injective homomorphisms). Thus the left commutative square is a pullback: let us assume that we have two homomorphisms: $f : N \rightarrow M' \otimes K$ and $g : N \rightarrow M'' \otimes K$ such that $(i' \otimes id_K) \circ f = (i'' \otimes id_K) \circ g$. Then one can easily see that $(p' \otimes id_K) \circ f = 0$ (since the far most vertical homomorphism is a monomorphism). Thus there exists a map $h : N \rightarrow (M' \cap M'') \otimes K$ such that $f = i_1 \circ h$. By commutativity of the left square and since $i'' \otimes id_K$ is a monomorphism it follows that $i_2 \circ h = g$. Uniqueness of h is a consequence of the fact that i_1 is an injection. The proposition follows since pullbacks in the category of modules along monomorphisms are given by intersection. \square

Furthermore, it is not hard to show that tensoring with a flat module preserves all finite limits. Indeed, tensoring with a flat module preserves equalisers and finite products (which in the category of \mathbf{R} -modules are equal to finite sums, which are preserved by the tensor functor since it has a right adjoint). In this chapter we show that there is a large class of modules for which the tensor product functor preserves arbitrary intersections. We will also construct examples of flat and faithfully flat modules without this property.

Let N' be a submodule of N , $i : N' \subseteq N$, and let M be an \mathbf{R} -module. Then the **canonical image** of $M \otimes N'$ in $M \otimes N$ is the image of $M \otimes N'$ under the map $id_M \otimes i$. It will be denoted by $\mathbf{im}(M \otimes N')$.

Definition 2.1.26

Let M, N be \mathbf{R} -modules, and let $(N_\alpha)_{\alpha \in I}$ be a family of \mathbf{R} -submodules of N . We say that a module M has the **intersection property with respect to N** if the homomorphism:

$$\mathbf{im}(M \otimes (\bigcap_{\alpha \in I} N_\alpha)) \longrightarrow \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha)$$

is an isomorphism for any family of submodules $(N_\alpha)_{\alpha \in I}$. We say that M has the **intersection property** if the above condition holds for any \mathbf{R} -module N .

Note that if M is flat then it has the intersection property if and only if the map $M \otimes (\bigcap_{\alpha \in I} N_\alpha) \longrightarrow \bigcap_{\alpha \in I} (M \otimes N_\alpha)$ is an isomorphism for any family $\{N_\alpha\}$ of submodules of a module N .

Proposition 2.1.27

The intersection property is closed under direct sums.

Proof: Let $X = \bigoplus_{i \in I} X_i$, be a direct sum of modules with the intersection property. We let $\pi_i : \bigoplus_{i \in I} X_i \rightarrow X_i$ be the canonical projection on the i -th factor and $s_i : X_i \rightarrow \bigoplus_{i \in I} X_i$ be the canonical section. Let $(N_\alpha)_{\alpha \in J}$ be a family of submodules of an \mathbf{R} -module N . Then we have a split epimorphism $(\bigoplus_{i \in I} \pi_i) \otimes id_N$ with a right inverse $(\bigoplus_{i \in I} s_i) \otimes id_N$. Also for each $\alpha \in J$ the map $(\bigoplus_{i \in I} \pi_i) \otimes id_{N_\alpha}$ is a split epimorphism with right inverse $(\bigoplus_{i \in I} s_i) \otimes id_{N_\alpha}$. Furthermore, we have a family of split epimorphisms, whose sections are jointly surjective:

$$\begin{array}{ccc} & \xleftarrow{s_i \otimes id_N} & \\ \mathbf{im}((\bigoplus_{i \in I} X_i) \otimes N_\alpha) & \xrightarrow{\pi_i \otimes id_N} & \mathbf{im}(X_i \otimes N_\alpha) \end{array}$$

They induce the following family of projections with jointly surjective sections:

$$\begin{array}{ccc} & \xleftarrow{s_i \otimes id_N} & \\ \bigcap_{\alpha \in J} \mathbf{im}((\bigoplus_{i \in I} X_i) \otimes N_\alpha) & \xrightarrow{\pi_i \otimes id_N} & \bigcap_{\alpha \in J} \mathbf{im}(X_i \otimes N_\alpha) \end{array}$$

For this let $x \in \bigcap_{\alpha \in J} \mathbf{im}((\bigoplus_{i \in I} X_i) \otimes N_\alpha)$. Then, for each $\alpha \in J$, there exists $y_\alpha \in \bigoplus_{i \in I} \mathbf{im}(X_i \otimes N_\alpha)$ such that $(\bigoplus_{i \in I} s_i \otimes id_N)(y_\alpha) = \sum_{i \in I} s_i \otimes id_N(y_\alpha) = x$. Since $\bigoplus_{i \in I} s_i \otimes id_N$ is a monomorphism we get $y = y_\alpha \in \bigcap_{\alpha \in J} \bigoplus_{i \in I} \mathbf{im}(X_i \otimes N_\alpha)$ for all $\alpha \in J$. It follows that:

$$\bigcap_{\alpha \in J} \mathbf{im}((\bigoplus_{i \in I} X_i) \otimes N_\alpha) \xrightarrow{\bigoplus_{i \in I} \pi_i \otimes id_N} \bigoplus_{i \in I} \bigcap_{\alpha \in J} \mathbf{im}(X_i \otimes N_\alpha)$$

is an isomorphism with inverse $\oplus_{i \in I} s_i \otimes id_N$. Now the proposition will follow from the commutativity of the diagram:

$$\begin{array}{ccc}
 \mathbf{im}((\oplus_{i \in I} X_i) \otimes (\bigcap_{\alpha \in J} N_\alpha)) & \longrightarrow & \bigcap_{\alpha \in J} \mathbf{im}((\oplus_{i \in I} X_i) \otimes N_\alpha) \\
 \downarrow \simeq & & \uparrow \simeq \\
 \mathbf{im}(\oplus_{i \in I} (X_i \otimes (\bigcap_{\alpha \in J} N_\alpha))) & & \oplus_{i \in I} \bigcap_{\alpha \in J} \mathbf{im}(X_i \otimes N_\alpha) \\
 \searrow = & & \nearrow \simeq \\
 & \oplus_{i \in I} \mathbf{im}(X_i \otimes (\bigcap_{\alpha \in J} N_\alpha)) &
 \end{array} \quad (2.1)$$

We will go around this diagram from the top left corner to the top right one and prove that all the maps on the way are isomorphisms. The first map is an isomorphism since tensor product commutes with colimits. Clearly the second map is an isomorphism as well. The bottom right arrow in (2.1) is an isomorphism since all X_i ($i \in I$) have the intersection property and we already showed that the last homomorphism is an isomorphism. \square

Proposition 2.1.28

The intersection property is stable under taking direct summands.

Proof: Let M be a direct summand in a module P which has the intersection property. Let M' be a complement of M in P . Then we have a chain of isomorphisms:

$$\begin{aligned}
 \mathbf{im}\left(M \otimes \bigcap_{\alpha \in J} N_\alpha\right) \oplus \mathbf{im}\left(M' \otimes \bigcap_{\alpha \in J} N_\alpha\right) &\simeq \mathbf{im}\left((M \oplus M') \otimes \left(\bigcap_{\alpha \in J} N_\alpha\right)\right) \\
 &= \bigcap_{\alpha \in J} \mathbf{im}\left((M \oplus M') \otimes N_\alpha\right) \\
 &\simeq \bigcap_{\alpha \in J} \mathbf{im}\left(M \otimes N_\alpha \oplus M' \otimes N_\alpha\right) \\
 &\simeq \bigcap_{\alpha \in J} \mathbf{im}\left(M \otimes N_\alpha\right) \oplus \bigcap_{\alpha \in J} \mathbf{im}\left(M' \otimes N_\alpha\right)
 \end{aligned}$$

Since every isomorphism in the above diagram commutes with projection onto the first and second factor the composition also does. Thus it is the direct sum of the two natural maps:

$$\mathbf{im}\left(M \otimes \bigcap_{\alpha \in J} N_\alpha\right) \rightarrow \bigcap_{\alpha \in J} \mathbf{im}(M \otimes N_\alpha),$$

$$\mathrm{im}\left(M' \otimes \bigcap_{\alpha \in J} N_\alpha\right) \rightarrow \bigcap_{\alpha \in J} \mathrm{im}(M' \otimes N_\alpha).$$

It follows that both maps are isomorphisms, hence both M and M' have the intersection property. \square

Let us note that \mathbf{R} itself has the intersection property (since $\mathbf{R} \otimes -$ is naturally isomorphic to the identity functor). Every free \mathbf{R} -module has the intersection property by Proposition 2.1.27 and by Proposition 2.1.28 every projective \mathbf{R} -module has it too. Under one of the following conditions on the ring \mathbf{R} : \mathbf{R} is a left noetherian local ring, or \mathbf{R} is a domain satisfying the strong rank condition, i.e. for any $n \in \mathbb{N}$, any set of $n + 1$ elements of \mathbf{R}^n is linearly dependent, then any finitely generated flat left \mathbf{R} -module is projective (see [Lam, 1999, Thm 4.38]). Thus it has the *intersection property*.

Definition 2.1.29

Let M be an R -module. It is a **Mittag–Leffler module** if for any family of R -modules M_λ ($\lambda \in \Lambda$) the canonical map

$$M \otimes \prod_{\lambda \in \Lambda} M_\lambda \rightarrow \prod_{\lambda \in \Lambda} (M \otimes M_\lambda), \quad m \otimes (m_\lambda)_{\lambda \in \Lambda} \mapsto (m \otimes m_\lambda)_{\lambda \in \Lambda}$$

where $m \in M$ and $m_\lambda \in M_\lambda$ for each $\lambda \in \Lambda$, is a monomorphism. The above condition is also called the *Mittag–Leffler condition*.

Proposition 2.1.30 ([Raynaud and Gruson, 1971, Cor. 2.1.7])

Let M be a Mittag–Leffler \mathbf{R} -module (where \mathbf{R} is a not necessarily commutative ring). For every projective filtered system of \mathbf{R}^{op} -modules (Q_r, u_{rs}) the canonical map $(\lim Q_r) \otimes_{\mathbf{R}} M \rightarrow \lim(Q_r \otimes_{\mathbf{R}} M)$ is injective. Furthermore, it is bijective if all the maps u_{rs} are injective.

Proof: The first part follows from the definition of Mittag–Leffler modules that we took. Note that Raynaud and Gruson use another equivalent definition of Mittag–Leffler modules (see [Raynaud and Gruson, 1971, Prop. 2.1.5]). The second claim follows from the snake lemma applied to the following exact diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\lim Q_r) \otimes M & \longrightarrow & Q_r \otimes M & \longrightarrow & (\lim(Q_r/Q_s)) \otimes_{\mathbf{R}} M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \lim(Q_r \otimes M) & \longrightarrow & Q_r \otimes M & \longrightarrow & \lim((Q_r/Q_s) \otimes M) \end{array}$$

where r is a fixed index. \square

Proposition 2.1.31

Let M be a flat Mittag-Leffler \mathbf{R} -module. Then M has the intersection property.

The above result follows from Proposition 2.1.30 by Raynaud and Gruson but we present here another proof.

Proof: Let N_α for $\alpha \in I$ be a family of submodules of an \mathbf{R} -module N . Let us consider the following diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & M \otimes (N / \bigcap_{\alpha \in I} N_\alpha) & \xrightarrow{i} & M \otimes (\prod_{\alpha \in I} N / N_\alpha) \\
 & & \downarrow G & & \downarrow f \\
 & & & & \prod_{\alpha \in I} (M \otimes (N / N_\alpha)) \\
 & & & & \downarrow g \\
 0 & \longrightarrow & (M \otimes N) / \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha) & \xrightarrow{j} & \prod_{\alpha \in I} (M \otimes N) / \mathbf{im}(M \otimes N_\alpha)
 \end{array}$$

Where i and j are the canonical embeddings:

$$i\left(m \otimes (n + \bigcap_{\alpha \in I} N_\alpha)\right) := m \otimes (n + N_\alpha)_{\alpha \in I}$$

and

$$j\left((m \otimes n) + \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha)\right) := (m \otimes n + \mathbf{im}(M \otimes N_\alpha))_{\alpha \in I}$$

for $m \in M$ and $n \in N$. While f sends $m \otimes (n_\alpha + N_\alpha)_{\alpha \in I}$ to $(m \otimes (n_\alpha + N_\alpha))_{\alpha \in I}$ and g is the canonical isomorphism. Note that $\mathbf{im}(gfi) \subseteq \mathbf{im}(j)$ and hence if M is a flat Mittag-Leffler module, then $G := gfi$ can be considered an embedding $G : M \otimes (N / \bigcap_{\alpha \in I} N_\alpha) \rightarrow (M \otimes N) / \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha)$. Hence we get the exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & \mathbf{im}(M \otimes (\bigcap_{\alpha \in I} N_\alpha)) & \rightarrow & M \otimes N & \longrightarrow & M \otimes (N / \bigcap_{\alpha \in I} N_\alpha) & \longrightarrow 0 \\
 & \downarrow H & & \downarrow = & & \downarrow G & \\
 0 \rightarrow & \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha) & \rightarrow & M \otimes N & \rightarrow & (M \otimes N) / \bigcap_{\alpha \in I} \mathbf{im}(M \otimes N_\alpha) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \text{coker}(H) & & 0 & & \text{coker}(G) &
 \end{array}$$

where H is the canonical embedding of $\text{im}(M \otimes (\bigcap_{\alpha \in I} N_\alpha))$ into the module $\text{im}(\bigcap_{\alpha \in I} (M \otimes N_\alpha))$ as submodules of $M \otimes N$. By the snake lemma we get the following short exact sequence:

$$0 = \ker(G) \rightarrow \text{coker}(H) \rightarrow 0$$

Thus the monomorphism H is onto. \square

There is also another class of modules with the intersection property. To define it we need the following

Definition 2.1.32

Let \mathbf{R} be a ring, M an \mathbf{R} -module and κ be a regular uncountable cardinal. A direct system C of submodules of M is said to be a κ -dense system in M if:

1. C is closed under unions of well-ordered ascending chains of length $< \kappa$,
2. every subset of M of cardinality $< \kappa$ is contained in an element of C .

Definition 2.1.33

Let \mathbf{R} , M and κ be as above. Then M is κ -projective if it has a κ -dense system C of $< \kappa$ -generated projective modules.

A module is flat Mittag-Leffler if and only if it is \aleph_1 -projective as is shown in [Herbera and Trlifaj, 2009, Thm. 2.9]. Thus we get the following

Corollary 2.1.34

Any \aleph_1 -projective module has the intersection property.

We would like now to recall a result of Raynaud and Gruson which is relevant.

Proposition 2.1.35 ([Raynaud and Gruson, 1971, Prop. 2.1.8])

Let M be a flat \mathbf{R} -module, such that for every finitely generated free \mathbf{R}^{op} -module L and for each $x \in L \otimes_{\mathbf{R}} M$, the set of submodules Q of L such that $x \in Q \otimes_{\mathbf{R}} M$ has a smallest element. Then M is a Mittag-Leffler module.

Now using the above result we obtain:

Corollary 2.1.36

A flat module has the intersection property if and only if it is Mittag-Leffler.

Proof: If a module is flat and Mittag-Leffler then it has the intersection property by Proposition 2.1.31. On the other hand if M is flat and has the intersection property then it satisfies the Mittag-Leffler condition by the preceding result of Raynaud and Gruson: take the set of submodules Q of L (where L and M are as in the previous proposition) such that for a given $x \in L \otimes_{\mathbf{R}} M$, $x \in Q \otimes_{\mathbf{R}} M$. Then the smallest such Q , due to the intersection property, is just the intersection of all such submodules Q . \square

Example 2.1.37 Let p be a prime ideal of \mathbb{Z} . Then $\bigcap_i p^i = \{0\}$. We let \mathbb{Z}_p denote the ring of fractions of \mathbb{Z} with respect to p . The \mathbb{Z} -module \mathbb{Z}_p is flat, and $\mathbb{Z}_p \otimes_{\mathbb{Z}} \bigcap_i p^i = \{0\}$. On the other hand $\bigcap_i \mathbb{Z}_p \otimes_{\mathbb{Z}} p^i \cong \mathbb{Z}_p$. By a similar argument \mathbb{Q} doesn't possess the intersection property, even though it is flat over \mathbb{Z} . The problem is that, the intersection property is stable under arbitrary direct sums but not under cokernels. Now it is easy to construct a faithfully flat module which does not have the intersection property. The \mathbb{Z} -modules $\mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Q}$ are the examples.

In the proof of Propositions 2.1.27 and 2.1.28 we showed that the intersection property is stable under split exact sequences. However, the above examples show that the intersection property is not stable under pure (exact) sequences [Lam, 1999, Def. 4.83], i.e. whenever $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a pure exact sequence and M has the intersection property then M'' might not have it. It is well known that if M'' is flat then $M' \subseteq M$ is pure [Lam, 1999, Thm 4.85], hence 2.1.37 is indeed a source of counter examples. However, we can show the following proposition:

Proposition 2.1.38

Let M' be a pure submodule of a module M with the intersection property. Then M' has the intersection property.

Proof: We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{im} (M \otimes (\bigcap_{\alpha \in I} N_{\alpha})) & \longrightarrow & M \otimes N & \xrightarrow{g} & M \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \longrightarrow 0 \\
 & & \nearrow & & \nearrow & & \nearrow \\
 0 & \longrightarrow & \text{im} (M' \otimes (\bigcap_{\alpha \in I} N_{\alpha})) & \xrightarrow{H} & M' \otimes N & \xrightarrow{f} & M' \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \longrightarrow 0 \\
 & & \downarrow H' & & \downarrow \parallel & & \downarrow \parallel \\
 & & \downarrow & & \downarrow & & \downarrow G' \\
 0 & \longrightarrow & \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) & \longrightarrow & M \otimes N & \longrightarrow & (M \otimes N) / \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) \longrightarrow 0 \\
 & & \nearrow & & \nearrow & & \nearrow \\
 0 & \longrightarrow & \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha}) & \longrightarrow & M' \otimes N & \longrightarrow & (M' \otimes N) / \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha}) \longrightarrow 0
 \end{array}$$

It easily follows that H' is a monomorphism. Let $x \in \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha})$. To prove that x is in the image of H' it is enough to show that it goes to 0 under f . Now since H is an isomorphism it goes to 0 under g and thus it belongs to the kernel of f . \square

Note that for pure submodules of flat Mittag-Leffler modules the above Proposition follows easily since they necessarily are flat Mittag-Leffler.

Proposition 2.1.39

Let M be a locally projective \mathbf{R} -module. Then M is a flat Mittag–Leffler module, so it has the intersection property.

Proof: We first show that M is flat. Let $i : N \subseteq N'$ be an \mathbf{R} -submodule. And let $\sum_i m_i \otimes n_i \in \ker id_M \otimes i$. Let $\pi : F \rightarrow M$ be an epimorphism, where F is a free module. Let us put $M_0 \subseteq M$ the submodule generated by $\{m_i\}$. Since M is locally-projective we have

$$\begin{array}{ccccc} 0 & \longrightarrow & M_0 & \xrightarrow{i_0} & M \\ & & & \searrow \exists h & \downarrow id_M \\ & & F & \xrightarrow{\pi} & M \longrightarrow 0 \end{array} \quad (2.2)$$

and we have $\pi \circ h|_{M_0} = id_M|_{M_0}$, hence $\pi \circ h(m_i) = m_i$. We have a commutative diagram:

$$\begin{array}{ccc} M \otimes N & \xrightarrow{id_M \otimes i} & M \otimes N' \\ h \otimes id_N \downarrow & & \downarrow h \otimes id_{N'} \\ F \otimes N & \xrightarrow{id_F \otimes i} & F \otimes N' \\ \pi \otimes id_N \downarrow & & \downarrow \pi \otimes id_{N'} \\ M \otimes N & \xrightarrow{id_M \otimes i} & M \otimes N' \end{array}$$

Now we have $(\pi \circ h) \otimes id_N(\sum_i m_i \otimes n_i) = \sum_i m_i \otimes n_i$ because $\sum_i m_i \otimes n_i \in \text{im}(i_0 \otimes id_N)$. Since $\sum_i m_i \otimes n_i \in \ker(id_M \otimes i)$ and $id_F \otimes i$ is a monomorphism (a free module is flat) thus $(h \otimes id_N)(\sum_i m_i \otimes n_i) = 0$ so

$$\sum_i m_i \otimes n_i = (\pi \otimes id_N) \circ (h \otimes id_N)(\sum_i m_i \otimes n_i) = 0$$

Hence $\ker id_M \otimes i = \{0\}$.

Now we show that M has the Mittag–Leffler property. Let $M_i, i \in I$ be a family of \mathbf{R} -modules and let us consider the canonical map

$$M \otimes \prod_i M_i \rightarrow \prod_i (M \otimes M_i)$$

Let $\sum_i m_i \otimes (m_i^j)_{j \in I}$ be in the kernel of this map, where $m_i \in M$ and $m_i^j \in M_j$. As before let M_0 be the submodule of M generated by all the elements m_i and we set π, h as in (2.2). We conclude as before with the commutative diagram:

$$\begin{array}{ccc}
M \otimes \prod_i M_i & \longrightarrow & \prod_i (M \otimes M_i) \\
h \otimes id_{\prod_i M_i} \downarrow & & \downarrow \prod_i h \otimes id_{M_i} \\
F \otimes \prod_i M_i & \twoheadrightarrow & \prod_i (F \otimes M_i) \\
\pi \otimes id_{\prod_i M_i} \downarrow & & \downarrow \prod_i \pi \otimes id_{M_i} \\
M \otimes \prod_i M_i & \longrightarrow & \prod_i (M \otimes M_i)
\end{array}$$

□

Theorem 2.1.40

Every flat \mathbf{R} -module M has the intersection property (or equivalently has the Mittag-Leffler property) if and only if for any exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

with M' , M projective, M'' flat and any family of submodules $(N_\alpha)_{\alpha \in I}$ of an \mathbf{R} -module N the sequence:

$$0 \rightarrow \bigcap_{\alpha} (M' \otimes N_\alpha) \rightarrow \bigcap_{\alpha} (M \otimes N_\alpha) \rightarrow \bigcap_{\alpha} (M'' \otimes N_\alpha) \rightarrow 0 \quad (2.3)$$

is exact.

Proof: Every flat module is a colimit of projective modules. Any colimit of projective modules can be computed as a cokernel of a map between projective modules, by [Mac Lane, 1998, Thm 1, Chap. V, §2]. So let M'' be a flat module and let

$$\mathcal{E} : 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence, where M' and M are projective modules. The extension \mathcal{E} is pure, since M'' is flat [Lam, 1999, Thm 4.85]. Let $(N_\alpha)_{\alpha \in I}$ be a family of submodules of an \mathbf{R} -module N . We have a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M' \otimes \bigcap_{\alpha} N_{\alpha} & \longrightarrow & M \otimes \bigcap_{\alpha} N_{\alpha} & \longrightarrow & M'' \otimes \bigcap_{\alpha} N_{\alpha} \longrightarrow 0 \\
& & \wr \downarrow & & \wr \downarrow & & \downarrow f \\
0 & \longrightarrow & \bigcap_{\alpha} (M' \otimes N_{\alpha}) & \longrightarrow & \bigcap_{\alpha} (M \otimes N_{\alpha}) & \longrightarrow & \bigcap_{\alpha} (M'' \otimes N_{\alpha}) \longrightarrow 0
\end{array}$$

The upper row is exact by purity of $M' \subseteq M$. Thus the lower row is exact if and only if the canonical map f is an isomorphism. □

The exactness of (2.3) is rather difficult to obtain but it might be useful once we

know that every flat module is Mittag-Leffler. For example, by the classical Bass results, every flat left module is projective if (and only if) the ring is left perfect.

Chapter 3

Lattices of quotients and subobjects

This chapter contains important lattice-theoretical results. We focus on lattices that appear in the theory of bi- or Hopf algebras and their (co)actions. We examine two properties: *completeness* and being an (dually) *algebraic lattice*¹. Completeness of many lattices ought to be known, though the other property seems not to be addressed by any previous studies. Note that in complete lattices the join (the meet) is uniquely determined by the meet (the join). Some of the lattice operations might seem counter-intuitive, since everything seems to be dual to the classical structures. For example the infimum of subalgebras is given by intersection while the supremum is the intersection of all subalgebras which contain both of them. However, the coideals of a coalgebra behave differently: the supremum of two coideals is their sum and their meet is the sum of all coideals contained in the intersection. The appearance of the *intersection property* shall be explained by Theorem 1.1.14. Let us note that generally we work over a commutative ring, though some of the results are only proved over a field. We begin by showing that a lattice of coideals of a coalgebra, ordered by inclusion, is complete. Then we recall the duality between the quotients of a coalgebra C and closed subalgebras of the topological algebra $A = C^*$. From this duality we derive that the lattice of coideal is dually algebraic (Proposition 3.1.15 on page 47), by showing that the lattice of coideals is dually isomorphic to the lattice of closed subalgebras of the dual algebra C^* . We also construct a simple example (Example 3.1.18 on page 48) of coalgebras (finite dimensional algebras) whose lattices of quotients are not distributive or not modular. It turns out that the lattice of subcoalgebras of a coalgebra over a field is algebraic: the meet is given by intersection and join is the sum (Proposition 3.1.20 on page 49). It turns out that subcoalgebras of a \mathbf{k} -coalgebra form also a dually algebraic lattice which is isomorphic to the lattice of closed ideals of the topological algebra $A = C^*$. From the decomposition theorem of commutative coalgebras over a field \mathbf{k} , we derive

¹Dually algebraic lattices are the ones which opposite lattices are algebraic.

a decomposition theorem for their lattices of subcoalgebras (Corollary 3.1.28 on page 51). Then we discuss the theory of subcomodules of a C -comodule over a ring R . We show that the lattice of subcomodules is complete and if the coalgebra C is a flat Mittag–Leffler module then it is also algebraic. The join is simply given by the sum, and in the latter case the meet (also the infinite one) is given by intersection. Our further results heavily depend on this statement. Wisbauer showed that the category of comodules is equivalent to the smallest Grothendieck subcategory of C^* -modules which contain C (with its natural C^* -modules structure) if and only if C is locally projective. In this case it follows that the lattice of subcomodules of a comodule M is isomorphic to the algebraic lattice of C^* -submodules of the C^* -module M .

In section 3.2 we analyse the lattice of quotients and substructures of bialgebras and Hopf algebras, including normal and conormal coideals. We also define generalised quotients and generalised subalgebras of bialgebras (Definition 3.2.5 on page 56), and we show that their lattices are complete (Proposition 3.2.6). This is important for the construction of the Galois correspondence for comodule algebras over bialgebras in the following chapter. The lattice of generalised subalgebras is algebraic by Proposition 3.2.7 (on page 57). Note that the lattice of generalised quotients and generalised subbialgebras are isomorphic in some cases, for example when the bialgebra is finite dimensional over a field (Theorem 4.6.14.6.1 on page 98)

We end the chapter with some important examples of the lattices of generalised quotients and generalised subalgebras. For the group Hopf algebra $k[G]$ we show that the lattice of generalised quotients is anti-isomorphic to the lattice of subgroups of the group G , while the lattice of Hopf algebra quotients is isomorphic to the lattice of normal subgroups (Proposition 3.2.9 on page 58). If the group is finite it follows that generalised quotients of $k[G]^*$ is anti-isomorphic to the lattice of subgroups (Proposition 3.2.10 on page 60). Both propositions we formulate using G -sets rather than subgroups. The reason is that we get an isomorphism rather than an anti-isomorphism, secondly and most importantly, G -sets appear very naturally in the Grothendieck approach to Galois theory: the equivalence of categories of G -sets and split algebras over a Galois extension of fields.

The final example is the lattice of generalised quotients of the enveloping algebra of a Lie algebra \mathfrak{g} . It turns out that the result is very similar to the case of a group algebra. Here the lattice of generalised quotients turns out to be anti-isomorphic to the lattice of Lie subalgebras of \mathfrak{g} , while the lattice of Hopf algebra quotients is anti-isomorphic to the lattice of Lie ideals of \mathfrak{g} .

In the last section we note that the lattice of H -comodule subalgebras of an H -comodule algebra A is algebraic if H is a flat Mittag–Leffler module (Proposition 3.3.1 on page 69).

We end the chapter with a helpful table which lists all the fifteen lattices that we discuss, with the list of properties we prove together with references to the statements in this chapter.

3.1 Coalgebras

Definition 3.1.1

Let C be an \mathbf{R} -coalgebra. A **coideal** I is a submodule of C such that C/I is a coalgebra and the natural epimorphism $C \rightarrow C/I$ is a map of coalgebras. Furthermore, we say that $I \subseteq C$ is a **right (left) coideal** if C/I is a right (left) comodule and the quotient map $C \rightarrow C/I$ is a map of right (left) C -comodules.

Note that [Brzeziński and Wisbauer](#) define coideals as kernels of a surjective coalgebra homomorphism and then they show that these two definitions are equivalent. Let us cite this result here:

Proposition 3.1.2 ([Brzeziński and Wisbauer \[2003\]](#))

Let C be an \mathbf{R} -coalgebra and let I be an \mathbf{R} -submodule of C . We let $\pi : C \rightarrow C/I$ be the quotient \mathbf{R} -module map. Then the following conditions are equivalent:

1. I is a coideal in the sense of Definition 3.1.1,
2. I is a kernel of a surjective coalgebra map,
3. $\Delta(I) \subseteq \ker(\pi \otimes \pi)$, and $I \subseteq \ker \epsilon$.

Furthermore, if $I \subseteq C$ is a pure submodule, then (i)-(iii) are equivalent to:

1. $\Delta(I) \subseteq I \otimes C + C \otimes I$.

If (i) holds then C/I is cocommutative provided C is.

Proof: For the proof see [[Brzeziński and Wisbauer, 2003](#), Prop. 2.4]. □

Proposition 3.1.3

For a coalgebra C over a commutative ring \mathbf{R} the set of all coideals, denoted by $\text{cold}(C)$, forms a complete lattice with inclusion as the order relation. The lattice operations in $\text{cold}(C)$ are given by

$$\begin{aligned} I_1 \vee I_2 &= I_1 + I_2 \\ I_1 \wedge I_2 &= \sum_{\substack{I \in \text{cold}(C) \\ I \subseteq I_1 \cap I_2}} I \end{aligned}$$

Proof: It is enough to show that any infinite suprema exist. Let $\pi_\lambda : C \rightarrow C_\lambda$ ($\lambda \in \Lambda$) be a family of coalgebra epimorphisms with kernels I_λ . Let us take $I = \sum_{\lambda \in \Lambda} I_\lambda$ and let $p : C \rightarrow C/I$ be the natural projection. By the previous proposition it is enough to show that $p \otimes p \circ \Delta(I) = 0$, since $I \subseteq \ker \epsilon$. Let us take $c \in I$. It is a sum $c = \sum_{\lambda \in \Lambda} c_\lambda$, where each $c_\lambda \in I_\lambda$ and only finitely many of them are non-zero. Then

$$p \otimes p \circ \Delta(c) = \sum_{\lambda \in \Lambda} p(c_{\lambda(1)})p(c_{\lambda(2)}) = 0$$

The last equality follows since p factorises through each p_λ and for each λ we have

$$p_\lambda(c_{\lambda(1)})p_\lambda(c_{\lambda(2)}) = 0$$

since I_λ is a coideal. □

If C is a \mathbf{k} -coalgebra then the dual space C^* is a unital algebra with the following multiplication (the *convolution product*):

$$f * g(c) := f(c_{(1)})g(c_{(2)}), \quad f, g \in C^*$$

The unit of the convolution product is the counit $\epsilon : C \rightarrow \mathbf{k}$. Let V be a \mathbf{k} -vector space and let V^* be its dual. For a subspace $W \subseteq V$ we let $W^\perp := \{f \in V^* : f|_W = 0\}$, while for $W \subseteq V^*$ we will write $W^\perp := \bigcap_{f \in W} \ker f$. Note that if V is a finite dimensional space then the above maps define a bijective correspondence (which reverses the inclusion order) between the subspaces of V and the subspaces of its dual.

Proposition 3.1.4

Let C be a coalgebra over a field \mathbf{k} . Then

1. $J \subseteq C$ is a right (left) coideal if and only if J^\perp is a right (left) ideal in C^* .
2. if $I \subseteq C^*$ is a right (left) ideal in C^* , then I^\perp is a right (left) coideal in C ;

Proof: See [Sweedler, 1969, Prop. 1.4.5]. □

It follows that for a finite dimensional coalgebra C the poset of right (left) coideals of C is anti-isomorphic to the poset of right (left) ideals of C^* .

Proposition 3.1.5

Let C be a coalgebra over a field \mathbf{k} . Then

1. $J \subseteq C$ is a coideal if and only if $J^\perp \subseteq C^*$ is a subalgebra.
2. if $S \subseteq C^*$ is a subalgebra of C^* , then S^\perp is a coideal of C ;

Proof: See [Sweedler, 1969, Prop. 1.4.6]. □

In the finite dimensional case we have the following result:

Proposition 3.1.6

Let C be a finite dimensional coalgebra over a field \mathbf{k} . Then we have a dual isomorphism of lattices:

$$(\text{cold}(C), \wedge, +) \cong (\text{Sub}_{\text{Alg}}(C^*), \cap, \vee)$$

and thus $\text{cold}(C)$ is a dually algebraic lattice.

Proof: The isomorphism of lattices is given by $\text{cold}(C) \ni I \mapsto I^\perp := \{f \in C^* : f|_I = 0\} \in \text{Sub}_{\text{Alg}}(C^*)$. We have $(I_1 + I_2)^\perp = I_1^\perp \cap I_2^\perp$ for $I_i \in \text{cold}(C)$ ($i = 1, 2$), but furthermore this formula works for infinite joins of coideals. From this we get

$$\begin{aligned}
 (I_1 \wedge I_2)^\perp &= \left(\sum_{\substack{I \in \text{cold}(C) \\ I \subseteq I_1 \cap I_2}} I \right)^\perp = \left(\bigcap_{\substack{I \in \text{cold}(C) \\ I \subseteq I_1 \cap I_2}} I^\perp \right) \\
 &= \left(\bigcap_{\substack{A \in \text{Sub}_{\text{Alg}}(C^*) \\ I_1^\perp \cup I_2^\perp \subseteq A}} A \right) \quad (\text{since } \text{cold}(C^*)^{op} \cong \text{Sub}_{\text{Alg}}(C^*)) \\
 &= I_1^\perp \vee I_2^\perp
 \end{aligned}$$

□

Thus $\text{Quot}(C)$ is an algebraic lattice (see Definition 1.1.9 on page 9) if C is finite dimensional. We are going to show that it is algebraic regardless of the dimension of C . Let us note that every complete upper subsemilattice of the lattice of subvector spaces of a finite dimensional vector space (like $\text{cold}(C)$ for a finite dimensional coalgebra C) is algebraic, since every vector subspace is a compact element of the lattice of subvector spaces.

Remark 3.1.7 *Every complete sublattice of the lattice of subspaces of a finite dimensional vector space is dually algebraic, since the lattice of subvector spaces of a finite dimensional vector space V is anti-isomorphic to the lattice of subvector spaces of the dual vector space V^* .*

The lattices of: \mathbf{k} -subcoalgebras, \mathbf{k} -subbialgebras, \mathbf{k} -subHopf algebras, as we will see later, are sublattices of the lattice of subvector spaces.

In order to show that the lattice of coideals is algebraic regardless of the dimension we will need some finer tools to study the dual algebra. The first of these is the fundamental theorem of comodules.

Theorem 3.1.8 (Fundamental Theorem of Comodules)

Let C be a coalgebra over a field \mathbf{k} and let M be a right C -comodule. Any element $m \in M$ belongs to a finite dimensional subcomodule.

The proof can be found in many text books: it follows from [Sweedler, 1969, Cor. 2.1.4] or is proved in [Dăscălescu et al., 2001, Thm. 2.1.7].

Definition 3.1.9

Let C be an \mathbf{R} -coalgebra. We let $\text{Quot}(C) = \{C/I : I \text{ is a coideal of } C\}$ with order relation $C/I_1 \succcurlyeq C/I_2 \Leftrightarrow I_1 \subseteq I_2$.

Clearly $\text{Quot}(C)$ is anti-isomorphic to the lattice $\text{cold}(C)$ and thus, by Proposition 3.1.3, it is a complete lattice.

We will now study the dual algebra C^* in some more detail. It turns out that it is a topological algebra. Let X and Y be non empty sets. The **finite topology** on the mapping space Y^X is the product topology when we view Y^X as a product of $Y_x := Y$ for $x \in X$, where each Y_x is regarded as a discrete space. A basis for open sets in this topology is given by the sets of the form

$$\mathcal{U}_{g, x_1, \dots, x_n} = \{f \in Y^X : f(x_i) = g(x_i), i = 1, \dots, n\}$$

where $g \in Y^X$ and $\{x_i : i = 1, \dots, n\} \subseteq X$ is a finite subset. Every open set is a union of open sets of this form. Now, if X and Y are \mathbf{k} -vector spaces then $\text{Hom}_{\mathbf{k}}(X, Y)$ is a subspace of Y^X and we will consider the topology induced by the finite topology of Y^X . This topology on $\text{Hom}_{\mathbf{k}}(X, Y)$ is also called the **finite topology**. The following theorem holds:

Theorem 3.1.10

Let V be a vector space. Then the maps $\text{Sub}(V) \ni W \mapsto W^\perp \in \text{Sub}(V^)$ and $\text{Sub}(V^*) \ni W \mapsto W^\perp \in \text{Sub}(V)$ form a Galois connection. Furthermore, the map $\text{Sub}(V) \ni W \mapsto W^\perp \in \text{Sub}(V^*)$ is a monomorphism and $W \in \text{Sub}(V^*)$ is closed in this Galois connection if and only if W is closed in the finite topology on $V^* = \text{Hom}_{\mathbf{k}}(V, \mathbf{k})$.*

Proof: The proof can be found in [Dăscălescu et al., 2001, Thm 1.2.6]. \square

Corollary 3.1.11

There is a bijection between subspaces of V and closed subspaces of V^ .*

In the following definition we restrict ourselves only to discrete topological fields.

Definition 3.1.12

Let \mathbf{k} be a field considered with the discrete topology.

- *Let V be a \mathbf{k} -vector space. It is called a **topological vector space** if it is given together with a topology such that the addition of vectors and the scalar multiplication are continuous operations.*
- *Let A be an \mathbf{k} -algebra. We say that A is a **topological algebra** if A is a topological vector space and the multiplication and the unit are continuous.*

Example 3.1.13 Let V be a vector space. Then V^* together with the finite topology is a topological vector space. See [Dăscălescu et al., 2001, Prop. 1.2.1].

Lemma 3.1.14

Let C be a \mathbf{k} -coalgebra. Then C^ together with the finite topology is a topological algebra.*

Proof: The open sets of the form $\mathcal{O}_V := \{f \in C^* : f|_V = 0\}$, where $V \subseteq C$ is a finite dimensional subspace, form a basis of neighbourhoods of 0 and thus it is enough to show that the preimage of an open set \mathcal{O}_V is open. For this let (f, g) be such that $f * g \in \mathcal{O}_V$. Using the *Fundamental Theorem of Comodules* 3.1.8 there exists V_r a right subcomodule of C such that $V \subseteq V_r$, such that $\dim V_r < \infty$, and a finite dimensional left subcomodule of C , denoted by V_l , such that $V \subseteq V_l$. Then $\mathcal{O}_{V_r} \times \mathcal{O}_{V_l}$ is an open neighbourhood of $(f, g) \in C^* \times C^*$ such that $\mathcal{O}_{V_r} * \mathcal{O}_{V_l} \subseteq \mathcal{O}_V$. That is, for $a \in \mathcal{O}_{V_r}$ and $b \in \mathcal{O}_{V_l}$ and $v \in V$ we have $\Delta(v) \in V_r \otimes C \cap C \otimes V_l = V_r \otimes V_l$ and hence $(a * b)(v) = a(v_{(1)})b(v_{(2)}) = 0$, and thus $a * b \in \mathcal{O}_V$. Since, \mathbf{k} is considered as a discrete space the unit map $\mathbf{k} \rightarrow C^*$ is continuous. This shows that C^* is a topological \mathbf{k} -algebra. \square

Theorem 3.1.15

Let C be a coalgebra over a field \mathbf{k} . Then the lattice $\text{Quot}(C)$ is algebraic.

Proof: First let us observe that $\text{Quot}(C)$ is isomorphic to the lattice of closed subalgebras of C^* . This follows from Lemma 3.1.14, Theorem 3.1.10 and Proposition 3.1.5. Let $X \subseteq C^*$ be a finite set. Then the smallest closed subalgebra of C^* which contains X (denoted by S_X) is a compact element of the lattice of closed subalgebras of C^* . For a closed subalgebra S we have $S = \bigvee_{\substack{X \subseteq S \\ X \text{ finite}}} S_X$. \square

Definition 3.1.16

Let (L, \vee, \wedge) be a lattice. We say that it is **modular** if for all $a, b, c \in L$:

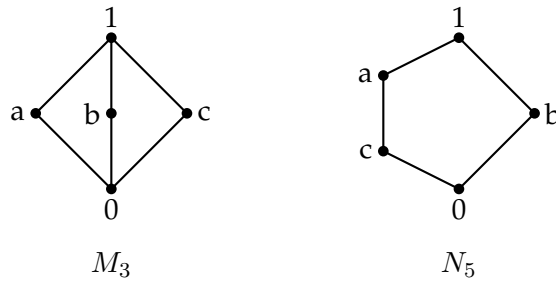
$$a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$$

A lattice L is **distributive** if for all $a, b, c \in L$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Let us note that if one of the distributive laws holds then the other is satisfied. If a lattice is distributive then it is modular. Let us introduce the following two lattices:



One can easily verify that M_3 is not distributive and N_5 is not modular.

Theorem 3.1.17

Let (L, \vee, \wedge) be a lattice.

1. The lattice L is modular if and only if L does not have a sublattice isomorphic to N_5 .
2. The lattice L is distributive if and only if L does not have a sublattice isomorphic to either N_5 or M_3 .

Proof: See [Roman, 2008, Thm. 4.7]. □

The lattices of coideals are in general neither modular nor distributive, since the lattice of submodules in general does not possess these properties. Below we present some examples.

Examples 3.1.18

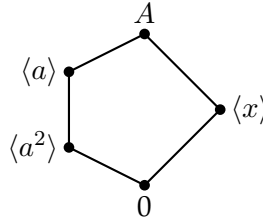
1. Let V be a vector space such that $\dim V \geq 2$. Then the lattice $\text{Sub}_{\text{Vect}}(V)$ is not distributive. It is modular, as every lattice of submodules is.
2. Let C be a coalgebra such that $C = C_0 \oplus k1$. For every $c \in C_0$ we set $\Delta(c) = c \otimes 1 + 1 \otimes c$, and $\Delta(1) = 1 \otimes 1$. The counit is set by $\epsilon(c) = 0$ for all $c \in C_0$ and $\epsilon(1) = 1$. The coalgebra C is cocommutative. Every subspace of $C_0 = \ker \epsilon$ is a coideal of C , i.e. $\text{cold}(C) = \text{Sub}_{\text{Vect}}(C_0)$. Thus the lattice of coideals is not distributive if $\dim C_0 \geq 2$.

Let us note that this coalgebra is dual to the commutative unital algebra $A = A_0 \oplus k1$, with unit 1 and such that for all $a, b \in A_0$ we have $a \cdot b = 0$ (with $\dim A_0 = \dim C_0$).

3. Let A be a finite dimensional commutative and unital algebra generated by two elements a and x with relations:

$$a^2x = a, \quad x^2 = 1, \quad a^4 = 1.$$

Then the lattice of subalgebras contains N_5 as a sublattice:



where $\langle y \rangle$ denotes the subalgebra generated by $y \in A$. Hence the lattice $\text{Quot}(A^*) \cong \text{Sub}_{\text{Alg}}(A)$ is not modular, where A^* is a (cocommutative) k -coalgebra since A is a finite dimensional (commutative) k -algebra.

Definition 3.1.19

Let C be an \mathbf{R} -coalgebra. An \mathbf{R} -coalgebra C' is called a **subcoalgebra** if

1. C' is a coalgebra,
2. C' is an \mathbf{R} -submodule of C , and
3. the inclusion map $C' \subseteq C$ is a coalgebra map.

The set of all subcoalgebras of a coalgebra C we will denote by $\text{Sub}_{\text{Coalg}}(C)$. It is a poset under the following relation: for subcoalgebras C' and C'' of a coalgebra C , $C' \preceq C''$ if and only if C' is a subcoalgebra of C'' .

If C' is a pure submodule (for example if \mathbf{R} is a field) then C' is a subcoalgebra of a coalgebra C if and only if $\Delta_C(C') \subseteq C' \otimes C'$, since by the purity $C' \otimes C' \subseteq C \otimes C$.

Theorem 3.1.20

Let C be a coalgebra over a field \mathbf{k} . Then the poset of subcoalgebras $\text{Sub}_{\text{Coalg}}(C)$ is a complete poset with lattice operations:

$$D_1 \vee D_2 := D_1 + D_2, \quad D_1 \wedge D_2 := D_1 \cap D_2$$

for $D_i \subseteq C$ ($i = 1, 2$) subcoalgebras of C . Furthermore, it is closed under arbitrary intersections and directed (set-theoretic) sums and thus it is an algebraic lattice.

Proof: Note that if D, D' are subcoalgebras of a coalgebra C then $D + D'$ is a subcoalgebra of C . Moreover, if \mathcal{O} is a family of subcoalgebras then $\sum_{D \in \mathcal{O}} D$ is a subcoalgebra of C . Thus $\text{Sub}_{\text{Coalg}}(C)$ is a complete lattice. Now let us show that if C_i (for $i \in I$) is a collection of subcoalgebras, then $\bigcap_{i \in I} C_i$ is a subcoalgebra. We have $\bigcap_{i \in I} C_i = \bigcap_{i \in I} (C_i^{\perp\perp}) = (\sum_{i \in I} C_i^{\perp})^{\perp}$. The first equality follows from Theorem 3.1.10 and Proposition 1.2.2(vi) and the second from Lemma 1.2.3. The sum $\sum_{i \in I} C_i^{\perp}$ is an ideal, since C_i^{\perp} are ideals of the algebra C^* . Thus $\bigcap_{i \in I} C_i = (\sum_{i \in I} C_i^{\perp})^{\perp}$ is a subcoalgebra in C . It follows that $\text{Sub}_{\text{Coalg}}(C)$ is a $\cap \bigcup$ -structure and by Theorem 1.1.14 it is an algebraic lattice. \square

Hence $\text{Sub}_{\text{Coalg}}(C)$ (for a \mathbf{k} -coalgebra C) is a sublattice of $\text{Sub}_{\mathbf{k}_{\text{Vect}}}(C)$ and thus by Remark 3.1.7, if C is finite dimensional this lattice is also dually algebraic. It turns out that this property holds for any \mathbf{k} -coalgebra.

Theorem 3.1.21

Let C be a \mathbf{k} -coalgebra. Then the lattice $\text{Sub}_{\text{Coalg}}(C)$ is anti-isomorphic to the lattice of closed ideals of the algebra C^* and thus it is a dually algebraic lattice.

Proof: The map $\text{Sub}_{\text{Coalg}}(C) \ni D \mapsto D^{\perp} \in \{I \in \text{Id}(A) : I\text{-closed}\}$ is a bijection by [Sweedler, 1969, Prop. 1.4.3] and Corollary 3.1.11. Since the lattice of closed ideals of a topological algebra is algebraic (with compact elements: closures of finitely generated ideals) the theorem follows. \square

Let C be a coalgebra over a field \mathbf{k} . For $c \in C$ there exists a smallest subcoalgebra, denoted by $C(c)$, such that $c \in C(c)$. Furthermore, by the *Fundamental Theorem of Coalgebras* it is finite dimensional.

Theorem 3.1.22 (Fundamental Theorem of Coalgebras)

Let C be a coalgebra over a field \mathbf{k} and let $c \in C$. Then there exists a finite dimensional subcoalgebra of C which contains c .

Proof: See [Dăscălescu et al., 2001, Thm 1.4.7]. □

Let $V \subseteq C$ be a subset. We let $C(V)$ be the smallest coalgebra which contains V . Clearly, $C(V) = C(\text{Span}(V))$, where $\text{Span}(V)$ denotes the vector subspace spanned by V . Furthermore, $C(V) = \sum_{v \in V} C(v)$. The compact elements of $\text{Sub}_{\text{Coalg}}(C)$ are precisely the subcoalgebras $C(V)$ where V is a finite subset of C , by Remark 1.1.15. Since these subcoalgebras are all finite dimensional and clearly all finite dimensional subcoalgebras are compact we conclude with

Proposition 3.1.23

A subcoalgebra B of a \mathbf{k} -coalgebra C (where \mathbf{k} is a field) is a compact element of $\text{Sub}_{\text{Coalg}}(C)$ if and only if $\dim B < \infty$.

Example 3.1.24 Let $C = C_0 \oplus \mathbf{k}1$ be the \mathbf{k} -coalgebra from Example 3.1.18(ii). Then a subspace $V \subseteq C$ is a subcoalgebra if and only if $1 \in V$. It is easy to observe that every such subspace is indeed a subcoalgebra. Now, let us assume that $D \subseteq C$ is a subcoalgebra. Take $d \in D$ and write it as $d = d_0 + \lambda 1$ where $d_0 \in C_0$ is non zero and $\lambda \in \mathbf{k}$. Let d_0^* be an element of C^* such that $d_0^*(d_0) = 1$ and $d_0^*(1) = 0$. Then $\Delta(d) = \lambda 1 \otimes 1 + d_0 \otimes 1 + 1 \otimes d_0 \in D \otimes D$. Now we apply $d_0^* \otimes \text{id}_C$ and we obtain: $1 \in D$. Thus we have an isomorphism of lattices:

$$\text{Sub}_{\text{Coalg}}(C) \cong \text{Sub}_{\text{Vect}}(C_0)$$

Hence $\text{Sub}_{\text{Coalg}}(C)$ is modular and it is not distributive if $\dim C_0 \geq 2$.

More can be said about the lattice of subcoalgebras of a cocommutative coalgebra. For this we need the following notions:

Definition 3.1.25

Let C be a coalgebra. It is called:

1. **simple** if it has no proper subcoalgebras, i.e. the only subcoalgebras are $\{0\}$ and C ;
2. **irreducible** if it has a unique simple subcoalgebra;
3. **pointed** if every of its simple subcoalgebras is one dimensional.

Since a sum of irreducible subcoalgebras which contain a common simple subcoalgebra is an irreducible subcoalgebra there exists maximal irreducible subcoalgebras. These are called **irreducible components**. An irreducible component that is pointed is called a **pointed irreducible component**.

For example the coalgebra $\mathcal{U}(\mathfrak{g})$, where \mathfrak{g} is a Lie algebra, is a pointed irreducible coalgebra, with the unique simple subcoalgebra $\mathbf{k}1 \subseteq \mathcal{U}(\mathfrak{g})$.

Let us note that every subcoalgebra contains a nontrivial simple subcoalgebra. By the *Fundamental Theorem of Coalgebras* (Theorem 3.1.22) it contains a finite dimensional subcoalgebra. If it is not simple, it contains a nontrivial subcoalgebra of smaller dimension. There must be a nonzero simple subcoalgebra by a finite induction. The *Fundamental Theorem of Coalgebras* shows also that simple coalgebras are all finite dimensional.

Theorem 3.1.26 ([Abe, 1980, Thm 2.4.7])

Let C be a coalgebra over a field \mathbf{k} . Then

1. an arbitrary irreducible subcoalgebra of C is contained in an irreducible component of C ;
2. a sum of distinct irreducible components of C is a direct sum;
3. if C is cocommutative then it is a direct sum of its irreducible components.

Let L, K be two lattices. Then $L \times K$ is a lattice with component wise operations and the order given by: $(l, k) \geq_{L \times K} (l', k')$ if and only if $l \geq_L l'$ and $k \geq_K k'$. The lattice $L \times K$ is called the product lattice of L and K .

Definition 3.1.27

Let L be a lattice. It is called *indecomposable* if it is not isomorphic to a product of two lattices.

As a corollary of Theorem 3.1.26(iii) we get.

Corollary 3.1.28

Let C be a cocommutative coalgebra over a field \mathbf{k} . Then the lattice $\text{Sub}_{\text{Coalg}}(C)$ has a direct product decomposition into indecomposable sublattices. Let C_i ($i \in I$) be the set of all irreducible components of C . Then the indecomposable components of $\text{Sub}_{\text{Coalg}}(C)$ are the sublattices $\text{Sub}_{\text{Coalg}}(C_i)$.

Proof: It only remains to show that for each irreducible component C_i of C the lattice $\text{Sub}_{\text{Coalg}}(C_i)$ is indecomposable. For this let $M_i \subseteq C_i$ be the unique simple subcoalgebra of C_i . Then for every $0 \neq D \subseteq C_i$, where D is a subcoalgebra, we have $M_i \leq D$. Now let us assume that $\text{Sub}_{\text{Coalg}}(C_i) \cong L \times K$, where L, K are sublattices. Since $\text{Sub}_{\text{Coalg}}(C_i)$ is bounded and complete so are the sublattices L and K . Then we must have $(0_K, 1_L) \geq M_i$ and $(1_K, 0_L) \geq M_i$ and thus $0_{K \times L} = (0_K, 1_L) \wedge (1_K, 0_L) \geq M_i$, hence $M_i = 0_{K \times L}$ which is a contradiction. \square

Let us now pass to C -comodules. The following theorem is an important step for us, since it will allow for implications when we mix algebraic structures like subalgebras with subcomodules (for example *generalised subalgebras* of bialgebras). It also will be used in the proof of the construction of a

Galois connection for H -extensions (Theorem 4.2.2 on page 78) and also when we compare our Galois connection with an earlier result of Schauenburg (Remark 4.2.3 on page 80).

Theorem 3.1.29

Let M be a (right) C -comodule, for a coalgebra C over a commutative ring \mathbf{R} . Then the poset of subcomodules of M , denoted by $\text{Sub}_{\text{Mod}^C}(M)$, is a complete lattice. For $N_i \in \text{Sub}_{\text{Mod}^C}(M)$, $i \in I$ we have:

$$\bigvee_{i \in I} N_i = \sum_{i \in I} N_i$$

Furthermore, if C is flat as an \mathbf{R} -module, then $N_1 \wedge N_2 = N_1 \cap N_2$ and the lattice of subcomodules of M is modular. The lattice $\text{Sub}_{\text{Mod}^C}(M)$ is algebraic if C is a flat Mittag-Leffler \mathbf{R} -module. In the latter case we thus have:

$$\bigwedge_{i \in I} N_i = \bigcap_{i \in I} N_i$$

for a family of subcomodules $N_i \subseteq M$, $i \in I$.

For $M = C$, for a coalgebra C over a field, one can prove this theorem in the same way as Theorem 3.1.15, since right coideals of C correspond to closed right ideals of C^* (by Theorem 3.1.10, Lemmas 3.1.14 and 3.1.4), which form an algebraic lattice. Let us note that if the ground ring is a field then the category of C -comodules is equivalent to the category of rational C^* -modules. Furthermore, a quotient module, and a submodule as well, of a rational module is rational by [Dăscălescu et al., 2001, Thm. 2.2.6]. Hence the lattice of subcomodules of a C -comodule M is isomorphic to the lattice of submodules of the rational module M with the induced C^* -module structure.

Proof of Theorem 3.1.29: First we note that if $(N_i)_{i \in I}$ is a family of subcomodules of a C -comodule M then

$$\bigvee_{i \in I} N_i = \sum_{i \in I} N_i \in \text{Sub}_{\text{Mod}^C}(M)$$

Thus the poset of subcomodules is a complete lattice. Let us assume that C is flat. Then $N_i \subseteq M$ is a C -subcomodule if $\delta(N_i) \subseteq N_i \otimes C \subseteq M \otimes C$, where $\delta : M \rightarrow M \otimes C$ is the C -comodule structure map. Then by Proposition 2.1.25 we have $\delta(N_1 \cap N_2) \subseteq (N_1 \otimes C) \cap (N_2 \otimes C) = (N_1 \cap N_2) \otimes C$. Thus $N_1 \cap N_2$ is a subcomodule and $N_1 \wedge N_2 = N_1 \cap N_2$.

Now, if C is a flat Mittag-Leffler module then for any family of its subcomodules $(N_i)_{i \in I}$, we have

$$\delta\left(\bigcap_{i \in I} N_i\right) \subseteq \bigcap_{i \in I} (N_i \otimes C) = \left(\bigcap_{i \in I} N_i\right) \otimes C$$

by Proposition 2.1.31 and thus $\bigcap_{i \in I} N_i$ is a C -subcomodule of M . Now it follows that $\text{Sub}_{\text{Mod } C}(C)$ is a $\cap \vec{\cup}$ -structure and thus is an algebraic lattice.

Brzeziński and Wisbauer, 3.13 show that the category of comodules is a Grothendieck category if C is flat as an \mathbf{R} -module. In Grothendieck categories the set of subobjects always forms a complete modular lattice. \square

Let us note that the lattice of subcomodules in general is not atomic. An atom of a lattice L , with the smallest element 0 , is an element $l > 0$ such that if $l' \in L$ is such that $l' < l$ then $l' = 0$. A lattice is called atomic if every element is a supremum of a subset of the set of atoms. Atoms of the lattice of subcomodules are exactly the simple subcomodules. Thus the lattice of subcomodules is atomic if and only if M is semisimple. For example let us consider C as a right C -comodule. Then

$$C \cong \bigoplus_{\substack{N \subseteq C \\ N\text{-simple subcomodule}}} E(N)$$

where $E(N)$ is the injective envelope of N (we refer to [Dăscălescu et al., 2001, Thm. 2.4.16] for injective envelopes in the categories of C -comodules). Since in general $N \subsetneq E(N)$ and N is the unique simple subcomodule which is contained in $E(N)$ we see that $E(N)$ cannot be a sum of simple subcomodules. This shows that the lattice of right coideals of C is atomic if and only if every simple right coideal of C is an injective C -comodule.

There is another case in which we can say something about the lattice of subcomodules. For this we need some module theoretic notions.

Definition 3.1.30

- Let M, N be \mathbf{R} -modules. We say that N is **generated** by M if there is an epimorphism $\bigoplus_{\lambda \in \Lambda} M \rightarrow N$ for some set Λ .
- We say that N is **subgenerated** by M if it is isomorphic to a submodule of an M -generated module.

The full subcategory of \mathbf{R} -modules which are subgenerated by M we denote by $\sigma_{\mathbf{R}}[M]$.

Definition 3.1.31 ([Wisbauer, 2004, sec. 4.6])

Let M be an \mathbf{R} -module. It is a **locally projective** if for any diagram with exact rows of the form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & M & & \\ & & & \searrow \exists h & \downarrow f & & \\ & & L & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where F is a finitely generated module, there exists $h : M \rightarrow L$, such that $g \circ h \circ i = f \circ i$.

If M is a right C -comodule, then it becomes a left C^* -module with the action $f \cdot m = ((id \otimes f) \circ \delta)(m) = m_{(0)}f(m_{(1)})$. It turns out that this construction is functorial and in some cases its image in the category of C^* -modules is the $\sigma_{C^*}[C]$ subcategory.

Theorem 3.1.32 ([Wisbauer, 2004, sec. 8.3])

Let C be an \mathbf{R} -coalgebra. Then the following conditions are equivalent:

1. $Mod^C \cong \sigma_{C^*}[C]$;
2. Mod^C is a full subcategory of ${}_{C^*}Mod$;
3. C is locally projective as a left \mathbf{R} -module;
4. every left C^* -submodule of C is a C -subcomodule.

Corollary 3.1.33

Let C be a coalgebra over a ring \mathbf{R} , such that C is locally projective left \mathbf{R} -module and let M be a (right) C -comodule. Then we have an isomorphism of posets $Sub_{Mod^C}(M) \cong Sub_{{}_{C^*}Mod}(M)$ and thus $Sub_{Mod^C}(M)$ is an algebraic lattice.

Proof: Since the category $\sigma_{C^*}[C]$ is closed under subobjects we have that the poset of C^* -submodules of M is equal to the poset of C^* -subgenerated submodules of M . Now the corollary follows from the above theorem. \square

3.2 Bialgebras and Hopf Algebras

Definition 3.2.1

Let B be a bialgebra and let H be a Hopf algebra, both over a commutative ring \mathbf{R} . Then:

1. a **biideal** of B is a kernel of a surjective morphism of bialgebras with domain B ; the poset of biideals (under inclusion) we will denote by $Id_{bi}(B)$;
2. a **subbialgebra** of B is a subalgebra and a subcoalgebra of B ; the poset of subbialgebras (under inclusion) we will denote by $Sub_{bi}(B)$;
3. a **Hopf ideal** of a Hopf algebra H is a kernel of a surjective morphism of Hopf algebras with domain H ; the poset of Hopf ideals (under inclusion) we will denote by $Id_{Hopf}(H)$;
4. a **subHopf algebra** (or Hopf subalgebra) K of a Hopf algebra H is a subbialgebra such that $S_H(K) \subseteq K$, where S_H is the antipode of H ; the poset of subHopf algebras (under inclusion) we will denote by $Sub_{Hopf}(H)$;

Let A be a \mathbf{k} -algebra and $B \subseteq A$ a subset of A . Then by $\langle B \rangle$ we will denote the smallest subalgebra of A which contains B . Let us note that if A is a bialgebra, then $\langle B \rangle$ is a subbialgebra. If A is a Hopf algebra then $\langle B \cup S(B) \rangle$ is the smallest Hopf subalgebra of A which contains B .

Proposition 3.2.2

Let H be a Hopf algebra and B a bialgebra over a ring \mathbf{R} .

1. The poset of biideals of B is a complete lattice. The join and meet are given by

$$I \vee J := I + J, \quad I \wedge J := \sum_{\substack{K \in \text{Id}_{bi}(B) \\ K \subseteq I \cap J}} K.$$

2. The poset of Hopf ideals of H form a complete lattice with operations:

$$I \vee J := I + J, \quad I \wedge J := \sum_{\substack{K \in \text{Id}_{Hopf}(H) \\ K \subseteq I \cap J}} K.$$

Furthermore, if \mathbf{R} is a field then

1. the poset of subbialgebras is an algebraic lattice, where meet and join have the form:

$$B_1 \vee B_2 := \langle B_1 + B_2 \rangle, \quad B_1 \wedge B_2 := B_1 \cap B_2$$

for $B_i \in \text{Sub}_{bi}(B)$ ($i = 1, 2$);

2. the poset of subHopf algebras is an algebraic lattice with operations:

$$H_1 \vee H_2 := \langle H_1 + H_2 \rangle, \quad H_1 \wedge H_2 := H_1 \cap H_2$$

for $H_i \in \text{Sub}_{bi}(H)$ ($i = 1, 2$).

Proof:

1. A sum of biideals (Hopf ideals) is a biideal (Hopf ideal, respectively) and thus it is their join in $\text{Id}_{bi}(B)$ ($\text{Id}_{Hopf}(H)$). Furthermore, both $\text{Id}_{bi}(B)$ and $\text{Id}_{Hopf}(H)$ are closed under arbitrary joins (sums of \mathbf{R} -submodules) and hence they are complete lattices. The formulas for the infimum follow from Remark 1.1.4.
2. The last two statements follow from Theorem 1.1.14, since subbialgebras and subHopf algebras (over a field) are closed under intersections and directed sums (see Theorem 3.1.20).

□

Let us note that the lattices of subbialgebras (subHopf algebras) are sublattices of the lattice of subspaces of the underlying vector space. Thus, by Remark 3.1.7, these lattices are also dually algebraic if the bialgebra (Hopf algebra) is finite dimensional.

Definition 3.2.3

Let $f : K \rightarrow H$ be a Hopf algebra homomorphism. Then:

1. f is called **normal** if for all $k \in K$ and $h \in H$ we have:

$$h_{(1)}f(k)S(h_{(2)}) \in f(K) \text{ and } S(h_{(1)})f(k)h_{(2)} \in f(K)$$

2. f is called **conormal** if for all $k \in \ker f$ we have:

$$k_{(2)} \otimes S(k_{(1)})k_{(3)} \in \ker f \otimes K \text{ and } k_{(2)} \otimes k_{(1)}S(k_{(3)}) \in \ker f \otimes K$$

A Hopf subalgebra $K \subseteq H$ is called **normal** if the inclusion map is normal. A Hopf ideal $I \subseteq H$ is called **normal** if the quotient map $H \rightarrow H/I$ is conormal.

Using Proposition 3.2.2 we obtain the following:

Corollary 3.2.4

Let H be a Hopf algebra over a field \mathbf{k} . Then the poset of normal Hopf subalgebras is an algebraic lattice. The poset of normal Hopf ideals is a complete lattice (for this it is enough that \mathbf{k} is a commutative ring).

Proof: The lattice of normal Hopf subalgebras is a lower subsemilattice of the lattice of Hopf subalgebras (which is closed under infinite meets). Furthermore, the lattice of normal Hopf ideals is an upper subsemilattice of the complete lattice of Hopf ideals (which is closed under infinite joins). Thus both lattices of normal subalgebras/ideals are complete. A directed sum of normal Hopf subalgebras is a normal Hopf subalgebra and also an intersection of two normal Hopf subalgebras is a normal Hopf subalgebra (see Theorem 3.1.20 on page 49). In this way normal subHopf algebras form a $\cap \overrightarrow{\cup}$ -structure. The poset of subHopf algebras is algebraic by Theorem 1.1.14. \square

Definition 3.2.5

1. A **generalised quotient** Q of a bialgebra B is a quotient by a right ideal coideal. The poset of generalised quotients will be denoted by $\text{Quot}_{\text{gen}}(B)$. The order relation of $\text{Quot}_{\text{gen}}(B)$ we will denote by \succ :

$$B/I \succ B/J \Leftrightarrow I \subseteq J$$

for $B/I, B/J \in \text{Quot}_{\text{gen}}(B)$.

2. A **generalised subalgebra** K of a bialgebra B is a left coideal subalgebra. The poset of generalised subalgebras will be denoted by $\text{Sub}_{\text{gen}}(B)$.

The poset $\text{Quot}_{\text{gen}}(B)$ is dually isomorphic to the poset of right ideals coideals of B , which will be denoted as $\text{Id}_{\text{gen}}(B)$. We define only the right version of generalised quotients and the left version of generalised subalgebras since we will consider right B -comodules (right B -comodule algebras).

Proposition 3.2.6

Let B be a bialgebra over a ring \mathbf{R} . Then the poset $\text{Quot}_{\text{gen}}(B)$ is a complete lattice.

Proof: There is a canonical isomorphism of posets $\text{Quot}_{\text{gen}}(B) \simeq \text{Id}_{\text{gen}}(B)^{\text{op}}$. The supremum in $\text{Id}_{\text{gen}}(B)$ is given by the sum of submodules, while the infimum is given by the formula:

$$I \wedge J = \sum_{\substack{K \subseteq I \cap J \\ K \in \text{Id}_{\text{gen}}(B)}} K \quad (3.1)$$

where $I, J \in \text{Id}_{\text{gen}}(B)$. □

Proposition 3.2.7

Let B be a bialgebra over a ring \mathbf{R} such that B is a flat Mittag-Leffler R -module. Then the poset $\text{Sub}_{\text{gen}}(B)$ is an algebraic lattice.

Proof: The poset $\text{Sub}_{\text{gen}}(B)$ is closed under (set theoretic) intersections (see the proof of Proposition 3.1.29), thus it is a complete lattice. It is also closed under directed sums and thus it is a $\cap \vec{\cup}$ -structure and by Theorem 1.1.14 it is an algebraic lattice. □

Quotients of $k[G]$ and its dual Hopf algebra

Before the next theorem we need the following

Definition 3.2.8

Let G be a group. A left (right) G -set S is a set together with a group homomorphism (anti-homomorphism respectively) $\phi : G \rightarrow \text{Bij}(S)$, where $\text{Bij}(S)$ is the group of bijections of S . We will write $gs = g(s) := \phi(g)(s)$ for left G -sets, and $sg := \phi(g)(s)$ for right G -sets.

A G -set S is called **transitive** if for any two $s, s' \in S$ there exists $g \in G$ such that $gs = s'$ ($sg = s'$ respectively).

A morphism of G -sets $f : S \rightarrow S'$ is a map of sets such that $f(gs) = gf(s)$ ($f(sg) = f(s)g$ respectively) for every $s \in S$.

The next two propositions describe the poset $\text{Quot}_{\text{gen}}(H)$ for $k[G]$ and its dual $k[G]^*$ (if G is a finite group). It turns out that they are isomorphic to the poset of isomorphism classes of transitive G -sets (which is anti-isomorphic to the poset of subgroups of G), while $\text{Quot}(H)$, the poset of Hopf algebra quotients, is isomorphic to the poset of quotient subgroups of G . In the Hopf-Galois theory considered in Chapter 4 we consider $\text{Quot}_{\text{gen}}(H)$ rather than $\text{Quot}(H)$. This parallels the use of subgroups of the Galois group in the classical Galois theory (or Grothendieck-Galois theory).

Proposition 3.2.9

Let $H = \mathbf{k}[G]$ where G is a group. Then

1. $\text{Quot}(\mathbf{k}[G]) \cong \text{Quot}(G)$, where $\text{Quot}(G)$ is the poset of quotient groups of G (which is anti-isomorphic to the poset of normal subgroups);
2. $\text{Quot}_{\text{gen}}(\mathbf{k}[G]) \cong \text{Quot}_{G\text{-set}}(G)$, where $\text{Quot}_{G\text{-set}}(G)$ is the poset of quotient G -sets of the free G -set G (which is anti-isomorphic to the poset of all subgroups of G).

Let us note that a transitive G -set S is of the form G/G_0 where G_0 is a subgroup of G , with the action induced by multiplication from the left for left G -sets and from the right for right G -sets.

We fix a notation. If C is a coalgebra, then a nonzero element $c \in C$ is called **group-like** if $\Delta(c) = c \otimes c$. The set of group-like elements we denote by $\mathbf{G}(C)$. If the base ring is a field, then the group-like elements are linearly independent. This fails for rings with idempotents: for example if the base ring \mathbf{R} contains an idempotent p then if $c \in C$ is a group-like element then pc is a group-like element as well. Furthermore, if B is a bialgebra over a field, then $\mathbf{G}(B)$ is a monoid with unit $1_B \in B$, since a product of two group-like element is group-like. If H is a Hopf algebra and $g \in H$ is a group-like element, then $S(g)$ is a group-like element, and moreover it is an inverse of g in $\mathbf{G}(H)$. In other words, $\mathbf{G}(H)$ is a group. It turns out that this gives rise to a functor from the category of bialgebras (Hopf algebras) to the category of monoids (groups). In the case of coalgebras it gives a functor from coalgebras to sets, adjoint to the functor $G \mapsto \mathbf{k}[G]$.

Proof of Proposition 3.2.9:

1. First let us note that if N is a normal subgroup, then $\mathbf{k}[G/N]$ is a quotient Hopf algebra of the Hopf algebra $\mathbf{k}[G]$ by the map induced by the projection $G \rightarrow G/N$. Now, let $\pi : \mathbf{k}[G] \rightarrow \mathbf{k}[G]/I$ be a Hopf algebra projection, i.e. let I be a Hopf ideal. Then there exists a set $G_0 \subseteq G$ such that $G' := \{\pi(g) : g \in G_0\}$ is a basis of $\mathbf{k}[G]/I$. It follows, that $\mathbf{G}(\mathbf{k}[G]/I) = G'$ is a quotient group of $G = \mathbf{G}(\mathbf{k}[G])$ via the map $g \mapsto \pi(g)$. Furthermore, since $\mathbf{k}[G]/I$ has a basis of group-like elements it is a group algebra, and $\mathbf{k}[G]/I \cong \mathbf{k}[G']$. Note that G' is a quotient group of G . This shows that the following two maps are inverses of each other:

$$\begin{array}{ccc}
 G & & \mathbf{k}[G] \\
 \text{Quot}(G) \ni p \downarrow & \xrightarrow{\quad \quad} & \downarrow \mathbf{k}[p] \in \text{Quot}(\mathbf{k}[G]) \\
 G' & & \mathbf{k}[G']
 \end{array}$$

$$\begin{array}{ccc}
& \mathbf{k}[G] & G \\
\text{Quot}(\mathbf{k}[G]) \ni \pi \downarrow & \dashrightarrow & \downarrow G(\pi) \in \text{Quot}(G) \\
& \mathbf{k}[G]/I & G(\mathbf{k}[G]/I)
\end{array}$$

2. Let $\mathbf{k}[G]/I$ be a quotient of $\mathbf{k}[G]$ by a coideal right ideal I . The set of group-like elements $\mathbf{G}(\mathbf{k}[G]/I)$ of the coalgebra $\mathbf{k}[G]/I$ is a $\mathbf{G}(\mathbf{k}[G])$ -set and hence a G -set, because $\mathbf{G}(\mathbf{k}[G]) = G$. Since the map $\mathbf{k}[G] \twoheadrightarrow \mathbf{k}[G]/I$ is an epimorphism it follows that:

- a) $\mathbf{k}[G]/I$ is spanned by group-like elements (as a \mathbf{k} -vector space),
- b) $\mathbf{G}(\mathbf{k}[G]/I)$ is a transitive (right) G -set.

On the other hand, if S is a (right) transitive G -set then $\mathbf{k}[S]$ is a right $\mathbf{k}[G]$ -module through the right G -action on S and a coalgebra quotient only if we set each $s \in S$ to be a group-like element, since $S \cong G/G_0$, where G_0 is a subgroup of G . Now, let us observe that these two constructions:

$$\text{Id}_{\text{gen}}(\mathbf{k}[G]) \ni I \mapsto \mathbf{G}(\mathbf{k}[G]/I) \in \text{Quot}_{G\text{-set}}(G)$$

and

$$\text{Quot}_{G\text{-set}}(G) \ni S \mapsto \ker(\mathbf{k}[G] \twoheadrightarrow \mathbf{k}[S]) \in \text{Id}_{\text{gen}}(\mathbf{k}[G])$$

are inverse to each other, and thus the claim follows. First, let us show that

$$I = \ker(\mathbf{k}[G] \twoheadrightarrow \mathbf{k}[\mathbf{G}(\mathbf{k}[G]/I)]).$$

This follows since we have a commutative diagram:

$$\begin{array}{ccc}
& \mathbf{k}[G] & \\
\swarrow & & \searrow \\
\mathbf{k}[\mathbf{G}(\mathbf{k}[G]/I)] & \xrightarrow[\alpha]{\cong} & \mathbf{k}[G]/I
\end{array}$$

where $\alpha(x) = x$ for all $x \in \mathbf{G}(\mathbf{k}[G]/I)$, and thus it is a monomorphism. It is an epimorphism by (G1). The remaining equality $S = \mathbf{G}(\mathbf{k}[S])$ is straightforward and it shows that $S = \mathbf{G}(\mathbf{k}[G]/\ker(\mathbf{k}[G] \twoheadrightarrow \mathbf{k}[S]))$

□

Proposition 3.2.10

Let G be a finite group. Then the following map

$$\text{Quot}_{G\text{-Set}}(G) \ni G/G_0 \mapsto \mathbf{k}[G_0]^* \in \text{Quot}_{\text{gen}}(\mathbf{k}[G]^*) \quad (3.2)$$

is an anti-isomorphism of posets.

Let us note that if G is finite then $\text{Quot}_{\text{gen}}(\mathbf{k}[G]) \cong \text{Sub}_{\text{gen}}(\mathbf{k}[G])$, by Theorem 4.6.1. In a consequence we have $\text{Quot}_{\text{gen}}(\mathbf{k}[G]) \cong \text{Quot}_{\text{gen}}(\mathbf{k}[G]^*)$ by Propositions 3.1.4 and 3.1.5. Hence the above proposition follows from the previous result. However, we present a direct construction.

Proof: Let us choose a basis of $\mathbf{k}[G]^*$ consisting of δ_g given by $\delta_g(h) = \delta_{g,h}$, where $\delta_{g,h}$ is the Kronecker symbol given by $\delta_{g,k} = \begin{cases} 1 & \text{iff } g=k \\ 0 & \text{otherwise} \end{cases}$. First let us observe that any right ideal I of $\mathbf{k}[G]^*$ must be generated by some subset of this basis. We have the equality $\delta_g \cdot \delta_h = \delta_{g,h} \delta_g$ for all $g, h \in G$. Let $\sum_{l=1}^n \lambda_l \delta_{g_l} \in I$ for some coefficients $\lambda_l \in \mathbf{k}$ and some $g_l \in G$ ($l = 1, \dots, n$). Then

$$\left(\sum_{l=1}^n \lambda_l \delta_{g_l} \right) \cdot \delta_{g_i} = \lambda_i \delta_{g_i}$$

and thus $\delta_{g_i} \in I$ for all $1 \leq i \leq n$ such that $\lambda_i \neq 0$. A right ideal I of $\mathbf{k}[G]^*$ is a coideal if and only if the set $M_I := \{g : \delta_g \notin I\}$ is a submonoid of G , i.e. it is closed under multiplication and contains the unit of G (since G is finite it is a subgroup of G). This is because I is a coideal of $\mathbf{k}[G]^*$ if and only if I^\perp is a subalgebra of $\mathbf{k}[G]$ (by Proposition 3.1.5(iii)) and M_I is a basis of I^\perp . On the other hand, a submonoid M of G defines a right ideal coideal I_M of $\mathbf{k}[G]^*$. The right ideal coideal I_M is spanned by all the δ_g for $g \notin M$. We have $I_M = \mathbf{k}[M]^\perp$, thus I_M is a coideal by Proposition 3.1.5 and it is a right ideal by Proposition 3.1.4, since $\mathbf{k}[M]$ is a right coideal subalgebra of $\mathbf{k}[G]$. We have a bijective correspondence between $\text{Id}_{\text{gen}}(\mathbf{k}[G]^*)$ and $\text{Sub}_{\text{group}}(G)$, which is given by $\text{Id}_{\text{gen}}(\mathbf{k}[G]^*) \ni I \mapsto M_I \in \text{Sub}_{\text{group}}(G)$ and $\text{Sub}_{\text{group}}(G) \ni G_0 \mapsto I_{G_0} \in \text{Id}_{\text{gen}}(\mathbf{k}[G]^*)$. Indeed, this is a pair of inverse bijections: for $G_0 \leq G$ a subgroup we have $M_{I_{G_0}} = G_0$ since all δ_g , for $g \in G$, are linearly independent. Moreover, for $I \in \text{Id}_{\text{gen}}(\mathbf{k}[G]^*)$, $I_{M_I} = I$, since we showed that I is spanned by the elements δ_g which form the basis of I_{M_I} (by definition of I_{M_I} its basis is $\{\delta_g : g \notin M_I\} = \{\delta_g : \delta_g \in I\}$). Now, the claim follows:

$$\text{Quot}_{G\text{-set}}(G)^{op} \cong \text{Sub}_{\text{group}}(G) \cong \text{Id}_{\text{gen}}(\mathbf{k}[G]^*)^{op} \cong \text{Quot}_{\text{gen}}(\mathbf{k}[G]^*)$$

This is indeed the map (3.2), since $\mathbf{k}[G]^*/I_{G_0} \cong \mathbf{k}[G_0]^*$. □

Quotients of $\mathcal{U}(\mathfrak{g})$

In this section we will consider the universal enveloping Lie algebra $\mathcal{U}(\mathfrak{g})$ where \mathfrak{g} is a finite dimensional Lie algebra. We will show how to construct all the generalised quotients and we shall prove the Poincaré–Birkhoff–Witt theorem for them. First let us state the Poincaré–Birkhoff–Witt theorem for universal enveloping algebras:

Theorem 3.2.11 (Poincaré–Birkhoff–Witt)

Let \mathfrak{g} be a finite dimensional \mathbf{k} -Lie algebra and $\{X_\lambda : \lambda \in \Lambda\}$ a totally ordered basis of \mathfrak{g} indexed by a set Λ . Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} . Then the set

$$\{1\} \cup \{X_{\lambda_1} \cdots X_{\lambda_k} \in \mathcal{U}(\mathfrak{g}) : X_{\lambda_1} \leq \dots \leq X_{\lambda_n}, \lambda_1, \dots, \lambda_n \in \Lambda, n \in \mathbb{N}^+\}$$

is a \mathbf{k} -linear basis of $\mathcal{U}(\mathfrak{g})$.

Now let us give a construction of a generalised quotient of $\mathcal{U}(\mathfrak{g})$. Let \mathfrak{h} be a \mathbf{k} -Lie subalgebra of \mathfrak{g} and let $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ be the \mathbf{k} -linear quotient map. Choose a linear basis C of the quotient space $\mathfrak{g}/\mathfrak{h}$. Define $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ to be the quotient of the free $\mathcal{U}(\mathfrak{g})$ -module $F(\mathbf{k}u \oplus \mathfrak{g}/\mathfrak{h})$ generated by the basis C by the following relation:

$$u \cdot X = \pi(X) \quad (3.3)$$

for all $X \in \mathfrak{g}$ such that $\pi(X) \in C$. It then follows that the above relation is satisfied for all $X \in \mathfrak{g}$. There exists a right $\mathcal{U}(\mathfrak{g})$ -module homomorphism:

$$\mathcal{Q}(\pi) : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{Q}(\mathfrak{g}/\mathfrak{h})$$

which is uniquely determined by $\mathcal{Q}(\pi)(X) = \pi(X)$ for $X \in \mathfrak{g}$, where $\pi(X) \in \mathfrak{g}/\mathfrak{h}$ is treated as an element of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ and $\mathcal{Q}(\pi)(1) = u$. The map $\mathcal{Q}(\pi)$ is well defined by the above Poincaré–Birkhoff–Witt Theorem and the following simple computation:

$$\begin{aligned} \mathcal{Q}(\pi)(XY - YX) &= \mathcal{Q}(\pi)(X)Y - \mathcal{Q}(\pi)(Y)X \\ &= \mathcal{Q}(\pi)(1)XY - \mathcal{Q}(\pi)(1)YX \\ &= \mathcal{Q}(\pi)(1)(XY - YX) \\ &= \mathcal{Q}(\pi)(1)([X, Y]) \\ &= \mathcal{Q}(\pi)([X, Y]) \end{aligned}$$

Note 3.2.12 Note that the following relation is satisfied in $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ (which is a consequence of 3.3):

$$\mathcal{Q}(\pi)(X)Y - \mathcal{Q}(\pi)(Y)X = \mathcal{Q}(\pi)([X, Y]) \quad (3.4)$$

for all $X, Y \in \mathfrak{g}$. This relation holds for every generalised quotient of $\mathcal{U}(\mathfrak{g})$. From this relation it follows that for any generalised quotient $p : \mathcal{U}(\mathfrak{g}) \rightarrow Q$ the subspace $\ker p \cap \mathfrak{g}$ is a \mathbf{k} -Lie subalgebra of \mathfrak{g} . This explains why we have assumed that $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.

Now we show, that $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is indeed a generalised quotient of $\mathcal{U}(\mathfrak{g})$.

Lemma 3.2.13

Let \mathfrak{g} be a finite dimensional \mathbf{k} -Lie algebra, \mathfrak{h} its Lie subalgebra, and $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ the \mathbf{k} -linear quotient map. Then $\mathcal{Q}(\pi) : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is a generalised quotient. The coalgebra structure of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is uniquely determined by:

$$\Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(\mathcal{Q}(\pi)(X)) = \mathcal{Q}(\pi)(X) \otimes u + u \otimes \mathcal{Q}(\pi)(X) \quad (3.5)$$

for all $X \in \mathfrak{g}$ and the requirement that $u \in \mathcal{Q}(\mathfrak{h})$ is a group-like element.

Proof: The map $\mathcal{Q}(\pi)$ is $\mathcal{U}(\mathfrak{g})$ -module surjection by the construction of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. We set the comultiplication of $\mathcal{U}(\mathfrak{h})$ by:

$$\Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(\mathcal{Q}(\pi)(X_0 \cdot \dots \cdot X_k)) := (\mathcal{Q}(\pi)(X_0) \otimes u + u \otimes \mathcal{Q}(\pi)(X_0)) \cdot (\Delta_{\mathcal{U}(\mathfrak{g})}(X_1 \cdot \dots \cdot X_k))$$

and $\Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(u) = u \otimes u$. It is well defined since the relation 3.3 is preserved:

$$\begin{aligned} \Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(u \cdot X) &:= (u \otimes u) \cdot (X \otimes 1 + 1 \otimes X) \\ &= (u \cdot X) \otimes u + u \otimes (u \cdot X) \\ &= \mathcal{Q}(\pi)(X) \otimes u + u \otimes \mathcal{Q}(\pi)(X) \\ &=: \Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(\mathcal{Q}(\pi)(X)) \end{aligned}$$

Clearly, $\mathcal{Q}(\pi) : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is a coalgebra map. Formula (3.5) uniquely determines the coalgebra structure of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ since for any Hopf algebra H and its generalised quotient $\pi : H \rightarrow Q$ we have:

$$\begin{aligned} \Delta_Q(\pi(h)k) &= \Delta_Q(\pi(hk)) \\ &= \pi \otimes \pi \circ \Delta_H(hk) \\ &= \pi \otimes \pi \circ (\Delta_H(h) \cdot \Delta_H(k)) \\ &= (\pi \otimes \pi \circ \Delta_H(h)) \cdot \Delta_H(k) \\ &= \Delta_Q(h) \cdot \Delta_H(k) \end{aligned}$$

for any $h, k \in H$. □

Theorem 3.2.14 (generalised Poincaré–Birkhoff–Witt)

Let $\mathfrak{h} \subseteq \mathfrak{g}$ and $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ be as above (in particular \mathfrak{g} is finite dimensional). Let B be a totally ordered basis of \mathfrak{g} . Then the set

$$\left\{ \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n \in \mathcal{Q}(\mathfrak{g}/\mathfrak{h}) : \begin{array}{l} Z_0, \dots, Z_n \in B \\ Z_0 \leq \dots \leq Z_n, \ n \in \mathbb{N} \\ \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n \neq 0 \end{array} \right\} \quad (3.6)$$

together with the single element u forms a \mathbf{k} -linear basis of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$.

Proof: Let us denote the free $\mathcal{U}(\mathfrak{g})$ -module on $\mathbf{k}u \oplus \mathfrak{g}/\mathfrak{h}$ by F . By definition of $\mathcal{U}(\mathfrak{g}/\mathfrak{h})$ we have an epimorphism $p : F \rightarrow \mathcal{U}(\mathfrak{g}/\mathfrak{h})$. Since F is just a sum of copies of $\mathcal{U}(\mathfrak{g})$, for which Theorem 3.2.11 holds, we can construct a map: $L : F \rightarrow F$ such that

$$\begin{aligned} L(\pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n) &= \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n \\ L(u \cdot Z_0 \cdot \dots \cdot Z_n) &= \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n \end{aligned}$$

where $Z_0, \dots, Z_n \in B$ and satisfy the condition $Z_0 \leq \dots \leq Z_n$, and $L(u) = u$. The kernel of $p : F \rightarrow \mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is generated as a $\mathcal{U}(\mathfrak{g})$ -submodule by the elements of the form: $u \cdot X - \pi(X)$ for $X \in B$ and hence it is spanned as a \mathbf{k} -vector space by the elements of the form: $u \cdot Z_0 \cdot \dots \cdot Z_n - \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n$ where $Z_0, \dots, Z_n \in B$ and $Z_0 \leq \dots \leq Z_n$. By definition of L we get $L|_{\ker(F \rightarrow \mathcal{U}(\mathfrak{g}))} = 0$. This proves that the elements (3.6) are linearly independent. Suppose on the contrary that there exists a linear relation:

$$\sum_{i=1}^N \lambda_i p(\pi(Z_0^i) \cdot Z_1^i \cdot \dots \cdot Z_{n_i}^i) = 0$$

where all the summands satisfy the conditions in 3.6. Then $\sum_i \lambda_i \pi(Z_0^i) \cdot Z_1^i \cdot \dots \cdot Z_{n_i}^i \in \ker p$ and hence $\sum_i \lambda_i \pi(Z_0^i) \cdot Z_1^i \cdot \dots \cdot Z_{n_i}^i = L(\sum_i \lambda_i \pi(Z_0^i) \cdot Z_1^i \cdot \dots \cdot Z_{n_i}^i) = 0$ in F . But the set:

$$\left\{ \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n \in F : \begin{array}{l} Z_0, \dots, Z_n \in B \\ Z_1 \leq \dots \leq Z_n, n \in \mathbb{N} \end{array} \right\}$$

is linearly independent by Theorem 3.2.11 and hence $\lambda_i = 0$ for $i = 1, \dots, N$. The claim that the set (3.6) and u span $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is straightforward. \square

Corollary 3.2.15

Let $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ be as above. Then the natural map $\mathfrak{g}/\mathfrak{h} \rightarrow \mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is injective.

Corollary 3.2.16

Let $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ be as above. Then $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is an irreducible pointed coalgebra.

Proof: The coalgebra $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is a quotient of the (irreducible) pointed coalgebra $\mathcal{U}(\mathfrak{g})$. Hence it is pointed. Furthermore, it is irreducible, since it has only one group-like element $\mathcal{Q}(\pi)(1)$. Let us assume that the following element is group-like:

$$x = \lambda_u u + \sum_{b \in B} \lambda_b b$$

where B is the set (3.6), only finitely many $\lambda_b \in \mathbf{k}$ are nonzero, and $\lambda_u \in \mathbf{k}$. All the elements of B belong to $\ker \epsilon_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}$. We get $x = \lambda_u u$, by computing $\epsilon \otimes id \circ \Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(x)$. We must have $x = u$ since distinct group-like elements are

linearly independent. \square

Now we are going to show that if \mathfrak{h} is a Lie ideal then the enveloping algebras $\mathcal{U}(\mathfrak{g}/\mathfrak{h})$ and $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ coincide, i.e. are isomorphic as generalised quotients of $\mathcal{U}(\mathfrak{g})$.

Proposition 3.2.17

Let \mathfrak{g} be a finite dimensional Lie algebra, \mathfrak{h} an ideal of \mathfrak{g} and $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ the quotient Lie algebra homomorphism. Then we have an isomorphism (of generalised quotients):

$$\begin{array}{ccc} & \mathcal{U}(\mathfrak{g}) & \\ \mathcal{Q}(\pi) \swarrow & & \searrow \mathcal{U}(\pi) \\ \mathcal{Q}(\mathfrak{g}/\mathfrak{h}) & \xrightarrow{\cong} & \mathcal{U}(\mathfrak{g}/\mathfrak{h}) \end{array} \quad (3.7)$$

where $\mathcal{U}(\pi)$ is the algebra homomorphism induced by the map π .

Proof: First we want to show that we have a natural epimorphism:

$$\mathcal{Q}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathcal{U}(\mathfrak{g}/\mathfrak{h})$$

For this we use the definition of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ as a quotient of a free $\mathcal{U}(\mathfrak{g})$ -module F generated by $\mathfrak{g}/\mathfrak{h}$. We have a $\mathcal{U}(\mathfrak{g})$ -module homomorphism $F(\pi) : F \rightarrow \mathcal{U}(\mathfrak{g}/\mathfrak{h})$ which sends a generator X of F , i.e. $X \in \mathfrak{g}/\mathfrak{h}$, to its image in $\mathcal{U}(\mathfrak{g}/\mathfrak{h})$, and $F(\pi)(u) = \mathcal{U}(\pi)(1)$. This map descends to the quotient $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. For this it is enough to check that:

$$\begin{aligned} F(\pi)(u \cdot X) &= F(\pi)(u) \cdot X \\ &= \mathcal{U}(\pi)(1) \cdot X \\ &= \mathcal{U}(\pi)(X) \\ &= F(\pi)(X) \end{aligned}$$

Hence we get the map $\mathcal{Q}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathcal{U}(\mathfrak{g}/\mathfrak{h})$. This map is an epimorphism since the diagram (3.7) commutes. It remains to show that it is a monomorphism. For this let us choose a totally ordered basis B of \mathfrak{g} such that a subset of B spans the Lie ideal \mathfrak{h} . Let us consider the associated Poincaré–Birkhoff–Witt basis of $\mathcal{U}(\mathfrak{g})$. The kernel of $\mathcal{U}(\pi)$ is spanned by all the PBW basis elements of the form $Z_1 \cdot \dots \cdot Z_n$ such that $Z_1 \leq \dots \leq Z_n$, $Z_1, \dots, Z_n \in B$ and there is at least one i such that $Z_i \in \mathfrak{h}$ (this follows from the Poincaré–Birkhoff–Witt theorem for $\mathcal{U}(\mathfrak{g}/\mathfrak{h})$). We claim that, for such an element, $\pi(Z_1) \cdot Z_2 \cdot \dots \cdot Z_n = 0$ as an element of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. We will prove this by induction under the smallest index i such that $Z_i \in \mathfrak{h}$. If $Z_1 \in \mathfrak{h}$ then clearly $\pi(Z_1) \cdot Z_2 \cdot \dots \cdot Z_n = 0$ in $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. Also if $Z_2 \in \mathfrak{h}$ then

$$\pi(Z_1) \cdot Z_2 \cdot \dots \cdot Z_n = \pi(Z_2) \cdot Z_1 \cdot \dots \cdot Z_n + \pi([Z_1, Z_2]) \cdot Z_3 \cdot \dots \cdot Z_n = 0$$

since \mathfrak{h} is a Lie ideal. If the induction hypothesis holds for i and the smallest index j such that $Z_j \in \mathfrak{h}$ is equal to $i + 1$ then by commuting Z_i and Z_{i+1} we get the claim under the induction hypothesis:

$$\begin{aligned} \pi(Z_1) \cdot Z_2 \cdot \dots \cdot Z_i \cdot Z_{i+1} \cdot \dots &= \pi(Z_1) \cdot Z_2 \cdot \dots \cdot Z_{i+1} \cdot Z_i \cdot \dots \\ &\quad + \pi(Z_1) \cdot Z_2 \cdot \dots \cdot [Z_i, Z_{i+1}] \cdot \dots \\ &= 0 \end{aligned}$$

□

An element p of a Hopf algebra H is called **primitive** if $\Delta(p) = p \otimes 1 + 1 \otimes p$. The set of primitive elements of a Hopf algebra H , which we denote by $P(H)$, forms a Lie algebra with the usual bracket $[p, q] = pq - qp$. Furthermore, $H \mapsto P(H)$ defines a functor from the category of Hopf algebras to the category of Lie algebras, that is if $f : H \rightarrow K$ is a Hopf algebra morphism then it restricts to a morphism of Lie algebras $P(f) : P(H) \rightarrow P(K)$. It is the right adjoint functor to \mathcal{U} which assigns the enveloping Hopf algebra of a Lie algebra discussed in Example 1.3.9. Let us also note that in characteristic zero we have $P(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$, while in characteristic p $X^p \in P(\mathcal{U}(\mathfrak{g}))$ for any $X \in \mathfrak{g}$.

There is another important consequence of the PBW basis for generalised quotients.

Proposition 3.2.18

Let \mathbf{k} be a field of characteristic 0. Let \mathfrak{g} be a finite dimensional \mathbf{k} -Lie algebra and \mathfrak{h} a Lie subalgebra of \mathfrak{g} . Then $P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h})) = \mathfrak{g}/\mathfrak{h}$.

Proof: The proof goes exactly the same way as for $\mathcal{U}(\mathfrak{g})$. We include it here for completeness. Let B be a totally ordered basis of \mathfrak{g} and let us consider the basis (3.6). For a basis element $b = \pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_m$ of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ we have:

$$\begin{aligned} \Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(b) &= b \otimes 1 + 1 \otimes b + \\ &\quad + \sum_{p=1}^{m-1} \sum_{\sigma - (p,m)\text{-shuffle}} \pi(Z_{\sigma(1)})Z_{\sigma(2)} \cdots Z_{\sigma(p)} \otimes \pi(Z_{\sigma(p+1)})Z_{\sigma(2)} \cdots Z_{\sigma(m)} \end{aligned} \quad (3.8)$$

where a (p, m) -shuffle σ is a permutation such that:

$$\sigma(1) < \sigma(2) < \dots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \dots < \sigma(m)$$

Note that since $Z_0 < \dots < Z_m$ both factors of the tensor $\pi(Z_{\sigma(1)})Z_{\sigma(2)} \cdots Z_{\sigma(p)}$ and $\pi(Z_{\sigma(p+1)})Z_{\sigma(2)} \cdots Z_{\sigma(m)}$ are basis elements (they are non zero, see (3.6)). By definition of the comultiplication $\Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}$ it follows that $\mathfrak{g}/\mathfrak{h}$ is a subset of the set of primitive elements of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. Now, let

$$u = \sum_{i=1}^l \alpha_i Z_i$$

where $\alpha_i \in \mathbf{k}$ and each $Z_i = \pi(Z_{i,1})Z_{i,2} \cdots Z_{i,m_i}$ is a basis element which belongs to (3.6). Let $v = \sum_{i:\alpha_i=1} Z_i \in \pi(\mathfrak{g})$. Then $u - v$ is also a primitive element. Hence without loss of generality we can assume that $m_i > 1$ for all $i = 1, \dots, l$. For such a primitive element u we will show that $u = 0$. By equation (3.8) and since u is a primitive element we have:

$$\begin{aligned} \sum_{i=1}^l \sum_{p=1}^{m_i-1} \sum_{\sigma - (p,m)\text{-shuffle}} \pi(Z_{i,\sigma(1)})Z_{i,\sigma(2)} \cdots Z_{i,\sigma(p)} \otimes \pi(Z_{i,\sigma(p+1)})Z_{i,\sigma(p+2)} \cdots Z_{i,\sigma(m_i)} \\ = \Delta_{\mathcal{Q}(\mathfrak{g}/\mathfrak{h})}(u) - u \otimes 1 - 1 \otimes u = 0 \end{aligned} \quad (3.9)$$

Let b and b' be basis elements which belong to (3.6) and let

$$I(b, b') := \left\{ (i, p, \sigma) : \begin{array}{l} \sigma \text{ is a } (p, m_i) \text{ shuffle,} \\ b = \pi(Z_{i,\sigma(1)})Z_{i,\sigma(2)} \cdots Z_{i,\sigma(p)}, \\ b' = \pi(Z_{i,\sigma(p+1)})Z_{i,\sigma(p+2)} \cdots Z_{i,\sigma(m_i)} \end{array} \right\}$$

Then the condition (3.9) is satisfied if and only if for any pair of basis elements b, b' we have:

$$\sum_{(i,p,\sigma) \in I(b,b')} \alpha_i = 0$$

We use the convention that the sum over an empty set is zero. Let us take $(i, p, \sigma), (j, q, \tau) \in I(b, b')$. Then we have:

$$\begin{aligned} \pi(Z_{i,\sigma(1)})Z_{i,\sigma(2)} \cdots Z_{i,\sigma(p)} &= b = \pi(Z_{j,\tau(1)})Z_{j,\tau(2)} \cdots Z_{j,\tau(q)} \\ \pi(Z_{i,\sigma(p+1)})Z_{i,\sigma(p+2)} \cdots Z_{i,\sigma(m_i)} &= b' = \pi(Z_{j,\tau(q+1)})Z_{j,\tau(q+2)} \cdots Z_{j,\tau(m_j)} \end{aligned}$$

It follows that $p = q$ and $m_i = m_j$ and for all $1 \leq t \leq m_i$:

$$Z_{i,\sigma(t)} = Z_{j,\tau(t)}$$

Now we have:

$$\#\{t : Z_{i,t} = X\} = \#\{t : Z_{j,t} = X\} \quad (3.10)$$

Now, if $Z_i \neq Z_j$ then there exists a minimal index t' such that $Z_{i,t'} \neq Z_{j,t'}$ and $Z_{i,t} = Z_{j,t}$ for all $t < t'$. Without loss of generality we can assume that $Z_{i,t'} < Z_{j,t'}$. Since $Z_{j,s} \leq Z_{j,s+1}$ for all s we have:

$$\begin{aligned} \#\{t : Z_{j,t} = Z_{i,t'}\} &= \#\{t : Z_{j,t} = Z_{i,t'} \text{ and } t < t'\} \\ &= \#\{t : Z_{i,t} = Z_{i,t'} \text{ and } t < t'\} \\ &< \#\{t : Z_{i,t} = Z_{i,t'}\} \end{aligned}$$

But this contradicts equation (3.10). Thus $Z_{i,t} = Z_{j,t}$ for all t and hence $Z_i = Z_j$. In consequence $i = j$. This shows that if $I(b, b')$ is non-empty, then there exist uniquely determined numbers i and p such that any element of

$I(b, b')$ looks like (i, p, σ) for some (p, m_i) -shuffle σ . Let us denote this number by $i_{b, b'}$. Equation (3.9) is equivalent to $\alpha_{i_{b, b'}} \cdot \#I(b, b') = 0$ for all pairs b, b' of basis elements of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$. Since $\#I_{b, b'}$ is a natural number and the characteristic of the field \mathbf{k} is zero we get $\alpha_i = 0$ for all i . Thus $u = 0$. This proves that all the primitive elements of $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ are those of $\mathfrak{g}/\mathfrak{h}$. \square

For a (one-sided) coideal subalgebra K of a Hopf algebra H we let $K^+ := K \cap \ker \epsilon$.

Proposition 3.2.19

Let \mathfrak{g} be a finite dimensional Lie algebra over a field \mathbf{k} of characteristic zero and let \mathfrak{h} be a Lie subalgebra. Then we have isomorphism of generalised quotients:

$$\begin{array}{ccc} & \mathcal{U}(\mathfrak{g}) & \\ p \swarrow & & \searrow Q(\pi) \\ \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g}) & \xleftarrow{\cong} & \mathcal{U}(\mathfrak{g}/\mathfrak{h}) \end{array} \quad (3.11)$$

where $p : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g})$ is the natural projection.

Proof: Let choose a Poincaré–Birkhoff–Witt basis for $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ as given in equation (3.6). Then we define a \mathbf{k} -linear map $\phi : \mathcal{Q}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g})$, by $\phi(\pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n) = p(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n$, where $\pi(Z_0) \cdot Z_1 \cdot \dots \cdot Z_n$ is a basis element which belongs to (3.6). Clearly, ϕ is a morphism of generalised quotients and hence, by commutativity of diagram (3.11), it is an epimorphism. Now let us consider $\phi|_{P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h}))} : P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h})) \rightarrow P(\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g}))$. We already know that $P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h})) = \mathfrak{g}/\mathfrak{h}$. Furthermore, $\mathfrak{g}/\mathfrak{h} \subseteq P(\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g}))$, which follows from the Poincaré–Birkhoff–Witt theorem for $\mathcal{U}(\mathfrak{g})$. It follows that $\phi|_{P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h}))} : P(\mathcal{Q}(\mathfrak{g}/\mathfrak{h})) \rightarrow P(\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g}))$ is a monomorphism. Now since $\mathcal{Q}(\mathfrak{g}/\mathfrak{h})$ is a pointed irreducible coalgebra by Corollary (3.2.16) it follows that ϕ is an injective homomorphism (by [Abe, 1980, Thm 2.4.11]). \square

Theorem 3.2.20

Let \mathfrak{g} be a finite dimensional Lie algebra over a field \mathbf{k} of characteristic 0. Then $\text{Quot}(\mathcal{U}(\mathfrak{g})) \cong \text{Quot}_{\text{Lie}}(\mathfrak{g})$, where $\text{Quot}_{\text{Lie}}(\mathfrak{g})$ denotes the poset of quotient Lie algebras of the Lie algebra \mathfrak{g} . Furthermore, there is an order reversing isomorphism $\text{Quot}_{\text{gen}}(\mathcal{U}(\mathfrak{g})) \cong \text{Sub}_{\text{Lie}}(\mathfrak{g})$, where $\text{Sub}_{\text{Lie}}(\mathfrak{g})$ is the poset of Lie subalgebras of the Lie algebra \mathfrak{g} .

Proof: We will show that the following two order reversing maps:

$$\begin{aligned} \text{Quot}_{\text{gen}}(\mathcal{U}(\mathfrak{g})) \ni Q &\mapsto \ker(\mathfrak{g} \rightarrow P(Q)) \in \text{Sub}_{\text{Lie}}(\mathfrak{g}) \\ \text{Sub}_{\text{Lie}}(\mathfrak{g}) \ni \mathfrak{h} &\mapsto \mathcal{Q}(\mathfrak{g}/\mathfrak{h}) \in \text{Quot}_{\text{gen}}(\mathcal{U}(\mathfrak{g})) \end{aligned}$$

are inverse bijections. Let $\pi : \mathcal{U}(\mathfrak{g}) \twoheadrightarrow Q$ be a generalised quotient. Since $\pi(\mathfrak{g})$ generates Q as a $\mathcal{U}(\mathfrak{g})$ -module we have an epimorphism $\phi : \mathcal{Q}(\pi(\mathfrak{g})) \rightarrow Q$ induced by the inclusion $\pi(\mathfrak{g}) \subseteq P(Q)$. The coalgebra $\mathcal{Q}(\pi(\mathfrak{g}))$ is pointed irreducible (see Corollary 3.2.16 on page 63). The map $\phi|_{P(\mathcal{U}(\pi(\mathfrak{g})))}$ is the inclusion $P(\mathcal{Q}(\pi(\mathfrak{g}))) = \pi(\mathfrak{g}) \subseteq Q$ when \mathbf{k} has characteristic zero (see Proposition 3.2.18 on page 65). It follows that ϕ is a monomorphism by [Abe, 1980, Thm 2.4.11]. Thus ϕ is an isomorphism. We get: $P(Q) = P(\mathcal{Q}(\pi(\mathfrak{g}))) = \pi(\mathfrak{g})$, and hence $P(Q)$ is indeed a quotient Lie algebra of \mathfrak{g} (or a quotient of \mathfrak{g} by a Lie subalgebra, see Note 3.2.12) and $\mathcal{Q}(P(Q)) = \mathcal{Q}(\pi(\mathfrak{g})) \cong Q$ as generalised quotients of $\mathcal{U}(\mathfrak{h})$. On the other hand if \mathfrak{h} is a quotient of \mathfrak{g} by a Lie subalgebra, then $\mathcal{Q}(\mathfrak{h}) \in \text{Quot}_{\text{gen}}(\mathcal{U}(\mathfrak{g}))$ and $P(\mathcal{Q}(\mathfrak{h})) = \mathfrak{h}$, since \mathbf{k} has characteristic zero (see Proposition 3.2.18).

Once we have shown the result for generalised quotients the claim for Hopf algebra quotients follows from Proposition 3.2.17 (on page 64). Note that if Q is a Hopf algebra quotient then the map $\mathfrak{g} = P(\mathcal{U}(\mathfrak{g})) \rightarrow P(Q)$ is an epimorphism of Lie algebras. Thus its kernel is a Lie ideal. \square

Quantum enveloping algebras

Heckenberger and Kolb classify all homogeneous right coideal subalgebras of a quantised enveloping algebra of a complex semisimple Lie algebra \mathfrak{g} . A (one-sided) coideal subalgebra is called homogeneous if it contains the Hopf subalgebra of all group-like elements. Interestingly the classification is based on the Weyl group W of \mathfrak{g} . The first hint that such a classification was possible was given by Kharchenko and Sagahon [2008]. They provided a complete classification of homogeneous right coideal subalgebras of the multiplier version of $U_q(\mathfrak{sl}_{n+1})$ (with q not a root of unity). This led to the conjecture that the number of homogeneous right coideal subalgebras of the Borel Hopf subalgebra of $U_q(\mathfrak{g})$ coincides with the order of the Weyl group of \mathfrak{g} . Kharchenko proved this conjecture for \mathfrak{g} of type A_n and B_n (Kharchenko and Sagahon [2008]; Kharchenko [2011]). The classification of right coideal subalgebras of $U(\mathfrak{g})$ done by Heckenberger and Kolb is based on the results of Heckenberger and Schneider [2009]. Let us note that in the classical case $\mathcal{U}(\mathfrak{h})$ is a Hopf subalgebra of $\mathcal{U}(\mathfrak{g})$ for any Lie subalgebra \mathfrak{h} of \mathfrak{g} (and $\mathcal{U}(\mathfrak{h}) = \mathcal{U}(\mathfrak{g})^{\text{co } \mathcal{Q}(\mathfrak{g}/\mathfrak{h})}$, which follows from Example 4.6.5). But in the quantum case this is no longer true: $\mathcal{U}_q(\mathfrak{h})$, when defined, might not be isomorphic to a Hopf subalgebra of $\mathcal{U}_q(\mathfrak{g})$. This actually led to the investigation of one sided coideal subalgebras, since there are analogs of $\mathcal{U}(\mathfrak{h})$ in $\mathcal{U}_q(\mathfrak{g})$ which are one sided coideal subalgebras (see Letzter [2002]).

Let us note that in the classical situation: for a separable field extension \mathbb{E}/\mathbb{F} which is a Hopf Galois with Hopf algebra H (over a subfield $\mathbf{k} \subseteq \mathbb{F}$, there exists an extension \mathbb{L} of \mathbf{k} such that $\mathbb{L} \otimes_{\mathbf{k}} H$ is isomorphic to a group Hopf algebra (see Greither and Pareigis [1987]). It seems that group-like elements

are essential in the commutative situation, though this might not be the case for non commutative extensions. Nevertheless let us note that the coideal right ideals which correspond (via Takeuchi correspondence, Theorem 4.6.1) to one sided coideal subalgebras classified by Heckenberger and Kolb do not contain any group-like elements other than the image of $1 \in H$. It is so since all the classified one sided coideal subalgebras contain the radical.

3.3 Comodule subalgebras

Let A be an \mathbf{R} -algebra. The poset of all subalgebras will be denoted by $\text{Sub}_{\text{Alg}}(A)$ and $\text{Sub}_{\text{Alg}}(A/B)$ is the interval $[B, A]$ in $\text{Sub}_{\text{Alg}}(A)$. Both lattices are algebraic. The compact elements of $\text{Sub}_{\text{Alg}}(A/B)$ are the finitely generated subalgebras, and the compact elements of $\text{Sub}_{\text{Alg}}(A/B)$ are the algebras which are generated by B and a finite set of elements of $A \setminus B$.

Let Q be a generalised quotient of a bialgebra H . Then A is a Q -comodule with the structure map $\delta_Q := (id \otimes \pi_Q) \circ \delta_A$, where $\pi_Q : H \rightarrow Q$ is the projection. If Q is a bialgebra quotient then A is a Q -comodule algebra. The poset of H -comodule subalgebras will be denoted by $\text{Sub}_{\text{Alg}^H}(A)$. Let us define the following interval in $\text{Sub}_{\text{Alg}^H}(A)$:

$$\text{Sub}_{\text{Alg}^H}(A/A^{coH}) := \{B \in \text{Sub}_{\text{Alg}^H}(A) : A^{coH} \subseteq B\}$$

The lattice of H -comodule subalgebras of A is an upper subsemilattice of $\text{Sub}_{\text{Alg}}(A)$ and thus it is complete. If H is flat as an \mathbf{R} -module, then for $S_i \in \text{Sub}_{\text{Alg}^H}(A)$ ($i = 1, 2$) we have $S_1 \wedge S_2 = S_1 \cap S_2$, which follows from Theorem 3.1.29.

Proposition 3.3.1

Let H be a bialgebra and let A be an H -comodule algebra such that H is a flat Mittag-Leffler \mathbf{R} -module. Then the posets $\text{Sub}_{\text{Alg}^H}(A)$ are algebraic lattices.

Proof: It follows from Theorem 3.1.29 that both $\text{Sub}_{\text{Alg}}(A)$ and $\text{Sub}_{\text{Alg}^H}(A)$ are $\cap \vec{\cup}$ -structures. \square

3.4 Summary

Let us bring together all the lattices considered:

POSET	OBJECTS	PROPERTIES
$\text{cold}(C)$	coideals of a coalgebra C	complete and dually algebraic (p.47)
$\text{Quot}(C)$	quotient coalgebras of a coalgebra C	algebraic (p.47)

$\text{Sub}_{\text{Coalg}}(C)$	subcoalgebras of a coalgebra C	algebraic (p.49) and dually algebraic (p.49) (over a field)
$\text{cold}_r(C)$	right coideals of a coalgebra C	complete, algebraic (p.52) (if C is a flat ML module)
$\text{cold}_l(C)$	left coideals of a coalgebra C	as above (p.52)
$\text{Sub}_{\text{Mod } C}(M)$	subcomodules of a right C -comodule M	as above (p.52)
$\text{Id}_{bi}(B)$	biideals of a bialgebra B	complete (p.55)
$\text{Sub}_{bi}(B)$	subbialgebras of a bialgebra B	algebraic (p.55) (over a field)
$\text{Id}_{\text{Hopf}}(H)$	Hopf ideals of a Hopf algebra H	complete (p.55)
$\text{Sub}_{\text{Hopf}}(H)$	subbialgebras of a Hopf algebra H	algebraic (p.55) (over a field)
$\text{Quot}(B)$	bialgebra (Hopf algebra) quotients of a bialgebra (Hopf algebra) B	complete (p.55)
$\text{Quot}_{gen}(B)$	quotients of a bialgebra (Hopf algebra) B by a right ideal coideal	complete (p.57)
$\text{Sub}_{gen}(B)$	left coideal subalgebras of a bialgebra (or a Hopf algebra) B	algebraic (p.57) (if B is a flat ML module)
$\text{Sub}_{\text{Alg}}(A)$	subalgebras of an algebra A	algebraic
$\text{Sub}_{\text{Alg}^H}(A)$	H -comodule subalgebras of an H -comodule algebra A	algebraic (p.69) (if H is a flat ML module)

In the above table an ML is a abbreviation for *Mittag-Leffler*.

Chapter 4

Galois Theory for Hopf–Galois extensions

In this chapter we introduce Galois theory for Hopf–Galois extensions. Let us note that the most common argument to use $\text{Quot}_{\text{gen}}(H)$ rather than $\text{Quot}(H)$ is that a noncommutative Hopf algebra might have too few (or even no non trivial) quotient Hopf algebras. As we noted before, there is another important argument, which shouldn't be missed. In the case of the group Hopf algebra $k[G]$ (and its dual $k[G]^*$ if G is finite) the poset of generalised quotients is isomorphic to the poset of isomorphism classes of transitive G -sets (which is anti-isomorphic to the poset of subgroups of G), while the Hopf quotients correspond to group quotients of G (normal subgroups). Since in Galois theory one considers the poset of subgroups of G (or more generally the category of G -sets, as in the Grothendieck approach to Galois Theory, see [Borceux and Janelidze \[2001\]](#)) it is more natural to use the poset of generalised quotients of Hopf algebras.

We begin with a section about the Schauenburg approach towards the Galois theory of Hopf Galois extensions of commutative rings. He considers a faithfully flat H -Galois extension of the commutative base ring and constructs a Hopf algebra $L(A, H)$ for which A is an $L(A, H)$ - H -bicomodule algebra and an $L(A, H)$ -Galois extension of the base ring. Then he constructs a Galois connection between H -comodule subalgebras of A and generalised quotients of $L(A, H)$. In the following section we introduce a more general construction of a Galois correspondence for H -extensions. Our result, in the following section, differs significantly from Schauenburg's. First of all, we construct a Galois connection between subalgebras (rather than H -comodule subalgebras) of an H -comodule algebra A and generalised quotients of H . Secondly, we drop the assumption that the coinvariants subalgebra must be equal to the commutative base ring. Importantly, we also leave the assumption that the extension is Hopf–Galois (Definition 1.3.13). We will require that both the

algebra and the Hopf algebra are flat Mittag-Leffler modules over the base ring. This extra module theoretical condition is always satisfied over a field. In this context we provide another, new, construction of the Galois connection. The formula obtained for the adjoint will be used later. It resembles, to some extent, the known formula for a Galois connection between left coideal subalgebras and generalised quotients of a Hopf algebra (see Theorem 4.6.1). Let us note that applying our construction to the left $L(A, H)$ -comodule algebra A , as considered by Schauenburg, one will get the same adjunction, which follows from the uniqueness of Galois correspondences (see Remark 4.2.3).

In section 4.3 (on page 86) we give two interesting results on closed elements, one for subalgebras (Theorem 4.3.1) and one for generalised quotients (Corollary 4.3.4). The first one shows that for an H -Galois extension satisfying some module theoretic assumptions, a subalgebra S is closed if and only if the following map is a bijection:

$$\text{can}_S : S \otimes_B A \longrightarrow A \square_{\psi(S)} H, \quad \text{can}_S(a \otimes b) = ab_{(1)} \otimes b_{(2)}$$

We make an attempt to show a similar result for generalised quotients. Assuming that the H -extension $A/A^{\text{co}H}$ has epimorphic canonical map, and satisfies some additional module theoretic conditions, we show (Corollary 4.3.4) that a generalised quotient $\pi : H \rightarrow Q$ is closed if the following map is a bijection:

$$\text{can}_Q : A \otimes_{A^{\text{co}Q}} A \longrightarrow A \otimes Q, \quad \text{can}_Q(a \otimes b) = ab_{(1)} \otimes \pi(b_{(2)})$$

In Theorem 4.7.1 we will show that for crossed products this is a sufficient and necessary condition. We also show, that for H -Galois extensions over a field, the above canonical map is an isomorphism if and only if Q is closed and the map $\delta_A \otimes \delta_A : A \otimes_{A^{\text{co}Q}} A \rightarrow (A \otimes H) \otimes_{A \otimes H^{\text{co}Q}} (A \otimes H)$ is an injective map (see Theorem 4.6.7 on page 102).

In section 4.4 (on page 89) we generalise Schauenburg's theory of admissible objects. We show that admissible subalgebras and admissible generalised quotients are in bijective correspondence for Hopf Galois extensions (under some module theoretic conditions). Let us note that we redefine admissibility of subalgebras of a comodule algebra. We show that our definition is equivalent with Schauenburg's when one considers faithfully flat H -Galois extensions of the base ring with the assumption that H has a bijective antipode (which is also considered by Schauenburg). Schauenburg obtained a 1-1 correspondence between admissible H -subcomodule algebras of A and admissible generalised quotients of $L(A, H)$. We show that, in our broader context, admissible subalgebras of A are in bijective correspondence with admissible generalised quotients of H . We also show that Schauenburg theory of admissible objects is a special case of our result (up to our stronger module theoretical assumptions).

4.1 Schauenburg's approach to Hopf-Galois extensions of commutative rings

Our approach to closed elements of the Galois connection between sub algebras of an H -extension and generalised quotients of H generalises the results of Schauenburg. His construction of the correspondence is different than ours and requires additional assumptions, nevertheless these assumptions and constructions allow one to use Hopf algebraic methods more extensively than in our approach. For the purpose of completeness and a better understanding we discuss here, in this preliminary section, the approach of Schauenburg. It is based on two articles [Schauenburg \[1996\]](#) and [Schauenburg \[1998\]](#). Schauenburg's construction was an adaptation of an earlier work of [van Oystaeyen and Zhang](#).

First let us note that Schauenburg's approach applies only to faithfully flat H -Galois extensions of commutative rings. In this section we will assume that A is an R -algebra which is faithfully flat as an R -module. Furthermore, we will assume that it possess an H -comodule algebra structure making it an H -Galois extension.

Before we start with the constructions we will introduce the following Sweedler-type notation:

$$\text{can}_H^{-1}(1 \otimes h) =: h_{[1]} \otimes h_{[2]} \in A \otimes A \quad (4.1)$$

where $h \in H$ and can_H is the *canonical* map (1.5). The above map satisfies the following relations:

$$(hk)_{[1]} \otimes (hk)_{[2]} = k_{[1]}h_{[1]} \otimes h_{[2]}k_{[2]} \quad (4.2a)$$

$$h_{[1]}h_{[2]} = \epsilon(h)1_A \quad (4.2b)$$

$$a_{(0)}(a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]} = 1_A \otimes a \quad (4.2c)$$

$$h_{[1]} \otimes (h_{[2]})_{(0)} \otimes (h_{[2]})_{(1)} = (h_{(1)})_{[1]} \otimes (h_{(1)})_{[2]} \otimes h_{(2)} \quad (4.2d)$$

$$(h_{[1]})_{(0)} \otimes h_{[2]} \otimes (h_{[1]})_{(1)} = (h_{(2)})_{[1]} \otimes (h_{(2)})_{[2]} \otimes S(h_{(1)}) \quad (4.2e)$$

where $h \in H$ and $a \in A$. Equation 4.2b is just a consequence of the following equality $\text{id}_A \otimes \epsilon \circ \text{can}_H(a \otimes b) = ab$; for (4.2c) it is enough to compute can_H of both sides and conclude that they are equal:

$$\begin{aligned} \text{can}_H(a_{(0)}(a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]}) &= a_{(0)}a_{(1)[1]}a_{(1)[2](0)} \otimes a_{(1)[2](1)} \\ &= a_{(0)} \otimes a_{(1)} \quad \text{by (4.1)} \\ &= \text{can}_H(1 \otimes a) \end{aligned}$$

where $a \in A$. Equation (4.2d) is a consequence of the fact that can_H is a right H -comodule map (where H -comodule structures on $A \otimes A$ and $A \otimes H$

are induced from the right tensor factor). Finally, the first one can now be checked in the following way:

$$\begin{aligned}
 \text{can}_H(b_{[1]}a_{[1]} \otimes a_{[2]}b_{[2]}) &= b_{[1]}a_{[1]}(a_{[2]})_{(0)}(b_{[2]})_{(0)} \otimes (a_{[2]})_{(1)}(b_{[2]})_{(1)} \\
 &= (b_{(1)})_{[1]}(a_{(1)})_{[1]}(a_{(1)})_{[2]}(b_{(1)})_{[2]} \otimes a_{(2)}b_{(2)} \quad \text{by (4.2d)} \\
 &= 1 \otimes ab \quad \text{by (4.2b)}
 \end{aligned}$$

Now we are ready to give the main Schauenburg's construction:

Proposition 4.1.1 ([Schauenburg, 1996, Prop. 3.5])

Let H be a Hopf algebra over a commutative ring \mathbf{R} and let A/\mathbf{R} be a faithfully flat H -Galois extension. Let $L(A, H) := (A \otimes A)^{\text{co}H}$, where $A \otimes A$ is considered with the codiagonal coaction of H :

$$A \otimes A \rightarrow A \otimes A \otimes H, \quad a \otimes b \mapsto a_{(0)} \otimes b_{(0)} \otimes a_{(1)}b_{(1)}$$

Then:

1. $L(A, H)$ is a subalgebra of $A \otimes A^{\text{op}}$;
2. the coalgebra structure is given by¹:

$$\begin{aligned}
 \Delta_{L(A, H)}(x \otimes y) &= x_{(0)} \otimes \text{can}_H^{-1}(1 \otimes x_{(1)}) \otimes y \\
 \epsilon_{L(A, H)}(x \otimes y) &= xy \in A^{\text{co}H} = \mathbf{R} \\
 S_{L(A, H)}(x \otimes y) &= y_{(0)} \otimes (y_{(1)})_{[1]}x(y_{(1)})_{[2]}
 \end{aligned}$$

The above structure turns $L(A, H)$ into an \mathbf{R} -Hopf algebra.

For the proof that this indeed defines a Hopf algebra structure on $(A \otimes A)^{\text{co}H}$ see [Schauenburg, 1996, Lemma 3.3 and Theorem 3.5]. Now, A becomes an $L(A, H)$ - H -biGalois extension. That means A/\mathbf{R} is:

1. a left $L(A, H)$ -comodule algebra which is an $L(A, H)$ -Galois extension of \mathbf{R} , with the $L(A, H)$ -comodule structure given by:

$$\delta_{L(A, H)} : A \rightarrow L(A, H) \otimes A, \quad \delta_{L(A, H)}(a) := a_{(0)} \otimes \text{can}_H^{-1}(1 \otimes a_{(1)})$$

2. a right H -comodule algebra which is an H -Galois extension of \mathbf{R} .

Moreover the two comodule structures: left $L(A, H)$ and right H -comodule commute (see the diagram below) making A an $L(A, H)$ - H -bimodule:

¹In the following formulas we use simple tensors, which is an abuse of notation, since the domain of the coalgebra structure maps is $(A \otimes A)^{\text{co}H}$, which might not be spanned by simple tensors. We do that for the sake of clarity.

$$\begin{array}{ccc}
A & \xrightarrow{\delta_H} & A \otimes H \\
\delta_{L(A,H)} \downarrow & & \downarrow \delta_{L(A,H)} \otimes id_H \\
L(A, H) \otimes A & \xrightarrow{id_{L(A,H)} \otimes \delta_H} & L(A, H) \otimes A \otimes H
\end{array}$$

We will use the following Sweedler notation for bicomodules:

$$\begin{aligned}
\delta_H(a) &:= a_{(0)} \otimes a_{(1)} \in A \otimes H \\
\delta_{L(A,H)}(a) &:= a_{(-1)} \otimes a_{(0)} \in L(A, H) \otimes A
\end{aligned}$$

where $a \in A$. We will show now that A is $L(A, H)$ -Galois. We will prove slightly more, since we will need this result later:

Lemma 4.1.2

Let H be a flat Hopf algebra over a commutative ring \mathbf{R} and let A/\mathbf{R} be a faithfully flat H -Galois extension. Finally, let $S \in \text{Sub}_{Alg}(A)$. Then the following map is a bijection:

$$can_S : A \otimes_S A \rightarrow (A \otimes_S A)^{co H} \otimes A, can_S(a \otimes_S b) = a_{(0)} \otimes_S (a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]} b$$

If we take $S = \mathbf{R}$ then can_S is the canonical map associated with the left $L(A, H)$ -comodule algebra A .

Proof: This is a consequence of the fundamental theorem for Hopf modules, but we will present a direct computation. The inverse of can_S is given by:

$$can_S^{-1} : (A \otimes_S A)^{co H} \otimes A \rightarrow A \otimes_S A, can_S^{-1}(a \otimes_S b \otimes c) = a \otimes_S bc$$

Indeed:

$$\begin{aligned}
can_S^{-1} \circ can_S(a \otimes b) &= a_{(0)} \otimes_S (a_{(1)})_{[1]} (a_{(1)})_{[2]} b \\
&= a_{(0)} \otimes_S \epsilon(a_{(1)}) b && \text{by (4.2b)} \\
&= a \otimes b
\end{aligned}$$

Now take $a \otimes b \otimes c \in (A \otimes_S A)^{co H} \otimes A$ (again we abuse the notation, since $(A \otimes_S A)^{co H}$ might not be generated by simple tensors):

$$\begin{aligned}
can_S \circ can_S^{-1}(a \otimes_S b \otimes c) &= can_S(a \otimes_S bc) \\
&= a_{(0)} \otimes_S (a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]} bc \\
&= a_{(0)} \otimes_S b_{(0)} (b_{(1)})_{[1]} (a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]} (b_{(1)})_{[2]} c && \text{by (4.2c)} \\
&= a_{(0)} \otimes_S b_{(0)} (a_{(1)} b_{(1)})_{[1]} \otimes (a_{(1)} b_{(1)})_{[2]} c && \text{by (4.2a)} \\
&= a \otimes_S b \otimes c
\end{aligned}$$

Where the last equation follows since $a \otimes b \in (A \otimes_S A)^{\text{co} H}$ and $1_{H[1]} \otimes 1_{H[2]} = 1_A \otimes 1_A$. \square

A very important property of this construction is that the pair

$$(L(A, H), \delta_{L(A, H)})$$

is the final object in the category (see [Schauenburg, 1996, Lem. 3.2]) in which objects are \mathbf{R} -modules M together with a right H -colinear map $\delta_M : A \rightarrow M \otimes A$, and where morphisms are commutative triangles:

$$\begin{array}{ccc} & A & \\ \delta_M \swarrow & & \searrow \delta_N \\ M \otimes A & \xrightarrow{f \otimes id_A} & N \otimes A \end{array}$$

Based on the above universal property Schauenburg shows the following

Proposition 4.1.3 ([Schauenburg, 1996, Prop. 3.5])

Let A be a faithfully flat H -Galois extension of the commutative base ring \mathbf{R} . Let B be a bialgebra which turns A into a B - H -biGalois extension. Then there exists a unique isomorphism $f : L(A, H) \rightarrow B$ compatible with the B -comodule and $L(A, H)$ -comodule structures on A .

Let us now state the Schauenburg construction of the Galois correspondence, which we are going to generalise in a subsequent section.

Proposition 4.1.4 ([Schauenburg, 1998, Prop. 3.2])

Let A/\mathbf{R} be a faithfully flat H -Galois extension and let us consider $L = L(A, H)$. Then there exists a Galois correspondence:

$$\text{Sub}_{\text{Alg}^H}(A) \rightleftarrows \text{Quot}_{\text{gen}}(L(A, H)) \quad (4.3)$$

where $\text{Sub}_{\text{Alg}^H}(A)$ is the poset of H -subcomodule algebras of A , a generalised quotient $Q \in \text{Quot}_{\text{gen}}(L(A, H))$ is mapped to ${}^{\text{co} Q} A := \{a \in A : \delta_{L(A, H)}(a) = 1_{L(A, H)} \otimes a\}$; while a subalgebra $S \in \text{Sub}_{\text{Alg}}(A)$ is mapped to $(A \otimes_S A)^{\text{co} H} \in \text{Quot}_{\text{gen}}(L(A, H))$.

Further, Schauenburg defines interesting classes of subalgebras and generalised quotients. Then he proves that these objects embed into closed elements of the above Galois correspondence. Later on we will amend this definition, so that it will work also for H -extensions with a noncommutative subalgebra of coinvariants.

Definition 4.1.5

1. Let C be a coalgebra and let $C \rightarrow D$ be a coalgebra quotient. We call D **left (right) admissible** if it is flat as an \mathbf{R} -module and C is right (left) faithfully flat as a D -comodule.

2. Let $S \in \text{Sub}_{\text{Alg}}(A)$, then S is called **left (right) admissible** if A is faithfully flat as a left (right) S -module.

Admissible objects are defined as the ones which are both left and right admissible.

The definition of admissibility for subalgebras will have to be altered in the next section. We will use the following result:

Proposition 4.1.6 ([Schauenburg, 1998, Prop. 3.5])

Let H be an \mathbf{R} -flat Hopf algebra with a bijective antipode. Then the map:

$$\text{Quot}_{\text{gen}}(H) \ni H/I \mapsto (H/I)^{\text{op}} := H/S_H(I) \in \text{Quot}_{\text{gen}}(H^{\text{op}})$$

is a bijection with inverse $\text{Quot}_{\text{gen}}(H^{\text{op}}) \ni H/I \mapsto (H/I)^{\text{op}} := H/S_{H^{\text{op}}}(I) \in \text{Quot}_{\text{gen}}(H^{\text{op}})$. The quotient $Q^{\text{op}} \in \text{Quot}_{\text{gen}}(H^{\text{op}})$ is right (left) admissible if and only if $Q \in \text{Quot}_{\text{gen}}(H)$ is left (right) admissible.

The main theorem concerning closed elements of (4.3) is the following:

Theorem 4.1.7 ([Schauenburg, 1998, Thm. 3.6])

Let H be a Hopf algebra with a bijective antipode and let A be a faithfully flat H -Galois extension of the base ring \mathbf{R} . Then the Galois correspondence (4.3) restricts to bijection between (left, right) admissible generalised quotients of $L(A, H)$ and H -subcomodule algebras S of A such that A is (left, right) faithfully flat over S .

4.2 Galois correspondence for H -extensions

We begin this section with a construction of a Galois connection between generalised subalgebras and generalised quotients of a bialgebra which is flat as an R -module. This Galois connection, as reported by Schauenburg is folklore. We put the proof here, since it is not written elsewhere in the generality that we will need.

Proposition 4.2.1

Let H be a bialgebra, which is flat over a commutative base ring R . Then the maps:

$$\text{Sub}_{\text{gen}}(H) \xrightleftharpoons[\phi]{\psi} \text{Quot}_{\text{gen}}(H) \quad (4.4)$$

where $\phi(Q) = H^{\text{co}Q}$, $\psi(K) = H/K^+H$ form a Galois connection.

Proof: First we show that the maps (4.4) are well defined and then we show that they form a Galois connection. Let $K \in \text{Sub}_{\text{gen}}(H)$ be a left coideal subalgebra, then K^+ is a coideal: let us consider $x = \Delta(k) - k \otimes 1 \in H \otimes K$, where $k \in K^+$. Since $\text{id} \otimes \epsilon(x) = 0$ and H is flat, we have $x \in H \otimes K^+$. Moreover $x - 1 \otimes k \in H^+ \otimes K^+$, because $\epsilon \otimes \text{id}(x) = k$. Thus $\Delta(k) - k \otimes 1 - 1 \otimes k \in H^+ \otimes K^+$.

Hence K^+H is a coideal and a right H -ideal. Now let $Q \in \text{Quot}_{\text{gen}}(H)$ and let π be the canonical surjection $\pi : H \rightarrow Q$. Then $H^{\text{co}Q}$ is a subalgebra: for $x, y \in H^{\text{co}Q}$ we have

$$\begin{aligned} (xy)_{(1)} \otimes \pi((xy)_{(2)}) &= x_{(1)}y_{(1)} \otimes \pi(x_{(2)}y_{(2)}) \\ &= x_{(1)}y_{(1)} \otimes \pi(x_{(2)})y_{(2)} \\ &= xy_{(1)} \otimes \pi(1)y_{(2)} \\ &= xy_{(1)} \otimes \pi(y_{(2)}) \\ &= xy \otimes \pi(1) \end{aligned}$$

It remains to show that it is a left coideal. For this let $x \in H^{\text{co}Q}$. One easily gets that

$$x_{(1)} \otimes x_{(2)(1)} \otimes \pi(x_{(2)(2)}) = x_{(1)} \otimes x_{(2)} \otimes \pi(1)$$

Hence $x_{(1)} \otimes x_{(2)} \in (H \otimes H)^{\text{co}Q} = H \otimes (H^{\text{co}Q})$, since H is a flat R -module. This shows that $H^{\text{co}Q}$ is a left H -comodule subalgebra.

Now let us show that these two maps indeed define a Galois connection. Let $Q = H/I \in \text{Quot}_{\text{gen}}(H)$ and let $\pi : H \rightarrow Q$ be the projection map. Then we want to show that $(H^{\text{co}Q})^+H \subseteq I$. For this let $\sum_i x_i y_i \in (H^{\text{co}Q})^+H$ be such that each $x_i \in (H^{\text{co}Q})^+$. Then we have:

$$\begin{aligned} \pi\left(\sum_i x_i y_i\right) &= \epsilon(x_{i(1)}y_{i(1)})\pi(x_{i(2)}y_{i(2)}) \\ &= \epsilon(x_{i(1)}y_{i(1)})\pi(x_{i(2)})y_{i(2)} \\ &= \epsilon(x_i y_{i(1)})\pi(y_{i(2)}) \\ &= 0 \end{aligned}$$

Now let us take a $K \in \text{Sub}_{\text{gen}}(H)$, and let $\pi_K : H \rightarrow H/K^+H$ be the natural projection. Then for every $k \in K$ we have:

$$k_{(1)} \otimes \pi_K(k_{(2)}) - k \otimes \pi_K(1) = k_{(1)} \otimes \pi_K(k_{(2)} - \epsilon(k_{(2)})1) = 0$$

Thus $K \subseteq H^{\text{co}H/K^+H}$, and indeed the maps (4.4) define a Galois connection. \square

Now we prove existence of the Galois correspondence for comodule algebras without the restriction of Schauenburg's approach. Afterwards we will also show how the result can be improved if the base ring happens to be a field. In this case the correspondence has a very similar form to the correspondence of the previous Proposition.

Theorem 4.2.2 (Galois correspondence for H -comodule algebras)

Let H be a bialgebra and let A/B be an H -extension over a commutative ring \mathbf{R} such that A and H are flat Mittag-Leffler \mathbf{R} -modules. Then we have a Galois connection:

$$\mathrm{Sub}_{Alg}(A/B) \xrightleftharpoons[\phi]{\psi} \mathrm{Quot}_{gen}(H) \quad (4.5)$$

where $\phi(Q) := A^{co Q}$ and ψ is given by the following formula:

$$\psi(S) = \bigvee \{Q \in \mathrm{Quot}_{gen}(H) : S \subseteq A^{co Q}\} \quad (4.6)$$

for $S \in \mathrm{Sub}_{Alg}(A/B)$.

Proof: We shall show that ϕ reflects all suprema: $A^{co \bigvee_{i \in I} Q_i} = \bigcap_{i \in I} A^{co Q_i}$. From the set of inequalities: $\bigvee_{i \in I} Q_i \succcurlyeq Q_j$ ($\forall j \in I$) it follows that $A^{co \bigvee_{i \in I} Q_i} \subseteq \bigcap_{i \in I} A^{co Q_i}$. Let us fix an element $a \in \bigcap_{i \in I} A^{co Q_i}$. We let I_i denote the coideal and right ideal such that $Q_i = H/I_i$. We identify $A \otimes I_i$ with a submodule of $A \otimes H$, which can be done under the assumption that A is flat over \mathbf{R} . We want to show that:

$$\begin{aligned} \forall_{i \in I} a \in A^{co Q_i} &\Leftrightarrow \forall_{i \in I} \delta(a) - a \otimes 1 \in A \otimes I_i \\ &\Leftrightarrow \delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i \\ &\Leftrightarrow a \in A^{co \bigvee_{i \in I} Q_i} \end{aligned}$$

The first equivalence is clear, the second follows from the equality: $\bigcap_{i \in I} A \otimes I_i = A \otimes \bigcap_{i \in I} I_i$ which holds since flat Mittag-Leffler modules have the intersection property (cf. Corollary 2.1.36). It remains to show that if $\delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i$ then $\delta(a) - a \otimes 1 \in A \otimes \bigwedge_{i \in I} I_i$. The other inclusion is clear, i.e. $A \otimes \bigwedge_{i \in I} I_i \subseteq A \otimes \bigcap_{i \in I} I_i$ (see formula (3.1) on page 57). We proceed in three steps. We first prove this for H , then for $A \otimes H$ and finally for a general comodule algebra A .

- (i) For $A = H$ this follows using the Galois connection (4.4). Let $h \in H$ be such that $\Delta(h) - h \otimes 1 \in H \otimes \bigcap_{i \in I} I_i$. Hence for all i we have $\Delta(h) - h \otimes 1 \in H \otimes I_i$, since H is a flat module, and as a consequence $h \in \bigcap_{i \in I} H^{co Q_i}$. Let $K = \bigcap_{i \in I} H^{co Q_i} \in \mathrm{Sub}_{gen}(H)$. Now by the Galois connection property for all $i \in I$ we have: $Q_i \leq H/H^{co Q_i} + H \leq H/K^+ H$. Thus $\bigvee_{i \in I} Q_i \leq H/K^+ H$ and as a consequence:

$$\bigcap_{i \in I} H^{co Q_i} = K \subseteq H^{co H/K^+ H} \subseteq H^{co \bigvee_{i \in I} Q_i}$$

Hence $\Delta(h) - h \otimes 1 \in H \otimes \bigwedge_{i \in I} I_i$.

- (ii) Now the proof for $A \otimes H$: let $x := \sum_{k=1}^n a_k \otimes h_k \in A \otimes H$ be such that $y := \sum_{k=1}^n a_k \otimes \Delta(h_k) - \sum_{k=1}^n a_k \otimes h_k \otimes 1_H \in A \otimes H \otimes \bigcap_{i \in I} I_i$. Let us write $y = \sum_{k=1}^m a'_k \otimes h'_k \otimes l'_k$ where $a'_k \in A$, $h'_k \in H$, $l'_k \in \bigcap_{i \in I} I_i$ for each $k = 1, \dots, m$. Since A is a flat Mittag-Leffler module, every finitely generated submodule of A is contained in a projective submodule [Herbera and Trlifaj, 2009, Thm 2.9]. Let A_0 be a projective submodule of A which contains both $\{a_k : k = 1, \dots, n\}$ and $\{a'_k : k = 1, \dots, m\}$. By projectivity of A_0 we can choose a "dual basis", i.e. a set of elements $e_j \in A_0$ and their "duals" $e^j \in A_0^*$, such that $e^j(a)$ is non zero only for finitely many $j \in J$ and $\sum_{j \in J} e_j e^j(a) = a$ for every $a \in A_0$. Using (i) above we get:

$$\sum_{k=1}^n e^j(a_k) \Delta(h_k) - e^j(a_k) h_k \otimes 1 = \sum_{k=1}^m e^j(a'_k) h'_k \otimes l'_k \in H \otimes \bigwedge_{i \in I} I_i$$

and by the dual basis property we conclude that:

$$a_k \otimes \Delta(h_k) - a_k \otimes h_k \otimes 1 \in A_0 \otimes H \otimes \bigwedge_{i \in I} I_i$$

It follows that $y \in A \otimes H \otimes \bigwedge_{i \in I} I_i$, as we claimed.

1. *The general case.* Let us observe that if $a_{(0)} \otimes a_{(1)} - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i$ then $a_{(0)} \otimes a_{(1)} \otimes a_{(2)} - a_{(0)} \otimes a_{(1)} \otimes 1 \in A \otimes H \otimes \bigcap_{i \in I} I_i$. Now by (ii): $a_{(0)} \otimes a_{(1)} \otimes a_{(2)} - a_{(0)} \otimes a_{(1)} \otimes 1 \in A \otimes H \otimes \bigwedge_{i \in I} I_i$. Computing $\text{id}_A \otimes \epsilon \otimes \text{id}_H$ we get $\delta(a) - a \otimes 1 \in A \otimes \bigwedge_{i \in I} I_i$.

The formula $\psi(S) = \bigvee \{Q \in \text{Quot}_{\text{gen}}(H) : S \subseteq A^{coQ}\}$ is an easy consequence of the Galois connection properties (cf. formula 1.3 on page 13). \square

Note that the above theorem also holds for left H -comodule algebras rather than right ones.

Remark 4.2.3 If B is commutative and H is a projective over the base ring then A/B is projective $(B \otimes H)$ -Hopf Galois extension by [Kreimer and Takeuchi, 1981, Thm 1.7]. Thus the above theorem applies as well as Schauenburg's result Proposition 4.1.4 (page 76). Note that Schauenburg constructs a Hopf algebra L such that A/B is an L - H -biGalois extension and he constructs a Galois connection between quotients of L and intermediate H -comodule subalgebras of A/B . The value of his approach is that the construction of L allows us to prove more about the correspondence (cf. [Schauenburg, 1998, Thm. 3.8]).

Let us note that if we apply the above theorem to the left $L(A, H)$ -comodule algebra we get the Schauenburg adjunction. Coinvariants of quotients of $L(A, H)$ are always H -subcomodule subalgebras, since A is an $L(A, H)$ - H -bicomodule algebra. Now, since we assume that H is a flat Mittag-Leffler module, the inclusion of H -comodule subalgebras of A into subalgebras $\text{Sub}_{\text{Alg}^H}(A) \subseteq \text{Sub}_{\text{Alg}}(A)$ preserves infinite intersections by Theorem 3.1.29. We conclude that both adjunctions coincide by the uniqueness of Galois connections (Proposition 1.2.2 on page 12).

Finally, let us note that in the Schauenburg context, the coinvariants of generalised quotients of H are always $L(A, H)$ -comodule subalgebras.

Theorem 4.2.4

Let A/B an H -extension over a base field \mathbf{k} , with the comodule structure map $\delta_A : A \rightarrow A \otimes H$ and let $S \in \text{Sub}_{\text{Alg}}(A/B)$. Then $\psi(S) = H/K_S^+ H$, where K_S is the smallest element of $\text{Sub}_{\text{gen}}(H)$ such that $\delta_A(S) \subseteq A \otimes K_S$, i.e.

$$K_S := \bigcap \{ K \in \text{Sub}_{\text{gen}}(H) : \delta_A(S) \subseteq A \otimes K \} \quad (4.7)$$

Proof: It is enough to show that $\phi : \text{Quot}_{\text{gen}}(H) \ni Q \mapsto A^{coQ} \in \text{Sub}_{\text{Alg}}(A/B)$ and $\psi' : \text{Sub}_{\text{Alg}}(A/B) \ni S \mapsto H/K_S^+ H \in \text{Quot}_{\text{gen}}(H)$ form a Galois connection, by Proposition 1.2.2(iii). Let us note that $K_S \in \text{Sub}_{\text{gen}}(H)$, as given by formula (4.7), is well defined since the infimum in $\text{Sub}_{\text{gen}}(H)$ is the set theoretic intersection, which follows from Proposition 3.1.29 on page 52. First we show that $S \subseteq A^{coH/K_S^+ H}$. Let $s \in S$ and let $\pi : H \rightarrow H/K_S^+ H$ be the canonical projection. Now, by definition of K_S , we have $\delta_A(s) \in A \otimes K_S$. We get the following equality in $A \otimes H/K_S^+ H$: $s_{(0)} \otimes \pi(s_{(1)}) = s_{(0)} \otimes \epsilon(s_{(1)})\pi(1) = s \otimes \pi(1)$. Thus $S \subseteq A^{coH/K_S^+ H}$. Now, let $Q \in \text{Quot}_{\text{gen}}(H)$ and let $K = H^{coQ}$. Then $\delta_A(A^{coQ}) \subseteq A \otimes K$, by commutativity of the following diagram:

$$\begin{array}{ccccc} A^{coQ} & \xrightarrow{\delta_A} & A \otimes H & \xrightarrow{id_A \otimes \Delta} & A \otimes H \otimes H \\ & & \downarrow id_A \otimes \pi & \delta_A \otimes id_H & \downarrow id_A \otimes id_H \otimes \pi \\ & & A \otimes Q & \xrightarrow{\delta_A \otimes id_Q} & A \otimes H \otimes Q \end{array}$$

For $a \in A^{coQ}$ we get $a_{(0)} \otimes a_{(1)} \otimes \pi(a_{(2)}) = a_{(0)} \otimes a_{(1)} \otimes \pi(1)$. Since we can choose all the elements $a_{(0)}$ to be linearly independent we get that $\delta_A(A^{coQ}) \subseteq A \otimes K$ and hence $K_{A^{coQ}} \subseteq K = H^{coQ}$. Now we get $H/K_{A^{coQ}}^+ H \supseteq H/K^+ H \supseteq Q$, where the last inequality follows from Theorem 4.6.1 on page 98. \square

Note that if H is finite dimensional and Q is a Hopf algebra quotient then $A^{coQ} = \delta_A^{-1}(A \otimes H^{coQ})$ (by [Schneider, 1992a, Thm. 1.7]). Let us note two similarities in this proof and the previous one. The existence of K_S is guaranteed by flatness and the Mittag-Leffler property of A (which over fields holds trivially). This is due to Proposition 2.1.30 (see also Proposition 2.1.35 on page 35). Also note that in both proofs we use the Galois connection 4.4 (which can be proven independently, cf. [Schauenburg, 1998, Theorem 3.10]).

Example 4.2.5 Let us take A to be the $\mathbf{k}[G]$ -comodule \mathbf{k} -algebra defined in Example 1.3.11(iii). The generalised quotients of $\mathbf{k}[G]$ are of the form $\pi :$

$\mathbf{k}[G] \rightarrow \mathbf{k}[G/G_0]$ for some subgroup G_0 of G by Proposition 3.2.9 (page 58). The correspondence:

$$\mathrm{Sub}_{Alg}(A/A_e) \xrightleftharpoons[\phi]{\psi} \mathrm{Quot}_{gen}(\mathbf{k}[G])$$

where $e \in G$ is the unit of G , is given by $\phi(\mathbf{k}[G/G_0]) = \sum_{g \in G_0} A_g$ and $\psi(B)$ corresponds to the smallest subgroup $G_0 \subseteq G$ such that $B \subseteq \sum_{g \in G_0} A_g$. We can see that in this example $\psi\phi = id_{\mathrm{Quot}_{gen}(\mathbf{k}[G])}$, but $\phi\psi$ might not be an identity morphism.

Remark 4.2.6 Let \mathbb{E}/\mathbb{F} be a finite field extension. Let $\mathrm{Sub}_{field}(\mathbb{F} \subseteq \mathbb{E})$ denote the lattice of subfields of a field \mathbb{E} containing a subfield \mathbb{F} and $\mathrm{Sub}_{ring}(\mathbb{F} \subseteq \mathbb{E})$ is the lattice of all intermediate subrings. There is the following diagram of Galois connections in which the upper Galois connection is the classical one and the lower one is the Galois connection (4.5).

$$\begin{array}{ccc} \mathrm{Sub}_{field}(\mathbb{E}/\mathbb{F}) & \xrightleftharpoons[\mathrm{Fix}]{\mathrm{Gal}} & \mathrm{Sub}(\mathrm{Gal}(\mathbb{E}/\mathbb{F})) \\ \downarrow & & \downarrow \cong \\ \mathrm{Sub}_{ring}(\mathbb{E}/\mathbb{F}) & \xrightleftharpoons[\phi]{\psi} & \mathrm{Quot}_{gen}(\mathbb{F}[\mathrm{Gal}(\mathbb{E}/\mathbb{F})]^*) \end{array}$$

where the right vertical map is the isomorphism of Proposition 3.2.10 (on page 60) and $(\mathrm{Fix}, \mathrm{Gal})$ is the classical adjunction of Galois theory of finite field extensions. Commutativity of this diagram follows from the formula:

$$\mathbb{E}^{co \mathbb{F}[G']^*} = \mathrm{Fix}(G')$$

It follows that ϕ factorises through the embedding $\mathrm{Sub}_{field}(\mathbb{F} \subseteq \mathbb{E}) \subseteq \mathrm{Sub}_{ring}(\mathbb{E}/\mathbb{F})$, and thus the only closed elements of the Galois connection (ϕ, ψ) in $\mathrm{Sub}_{ring}(\mathbb{E}/\mathbb{F})$ are the ones coming from closed elements of the lattice $\mathrm{Sub}_{field}(\mathbb{E}/\mathbb{F})$. Normal subgroups of the Galois group correspond to conormal quotients of H , i.e. quotients by a normal ideal, and also to normal subextensions.

Example 4.2.7 Let q be a primitive fourth root of unity and let $\mathbf{k} = \mathbb{Q}[q]$. Let A be the \mathbf{k} -algebra generated by J, X, Z subject to the relations: $XZ = qZX$, $X^4 = Z^{16} = 1$, $J^2 = -1$ (J commutes with both X and Z). We let $Y = Z^4$, and let B be the subalgebra generated by X and Y . Now let us construct the following Hopf algebra H . As an algebra it is generated by the elements c, s, t which satisfy the following relations:

$$\begin{aligned} c^2 + s^2 &= 1, & cs &= 0, \\ ct &= tc, & ts &= -st \\ t^2 &= 1 \end{aligned}$$

The coalgebra structure is given by:

$$\begin{aligned}\Delta(c) &= c \otimes c - s \otimes s \\ \Delta(s) &= s \otimes c + c \otimes s \\ \Delta(t) &= t \otimes t\end{aligned}$$

Coassociativity of the comultiplication is a straightforward calculation:

$$\begin{aligned}\Delta \otimes id \circ \Delta(c) &= c \otimes c \otimes c - s \otimes s \otimes c - s \otimes c \otimes s - c \otimes s \otimes s \\ id \otimes \Delta \circ \Delta(c) &= c \otimes c \otimes c - c \otimes s \otimes s - s \otimes c \otimes s - s \otimes s \otimes c\end{aligned}$$

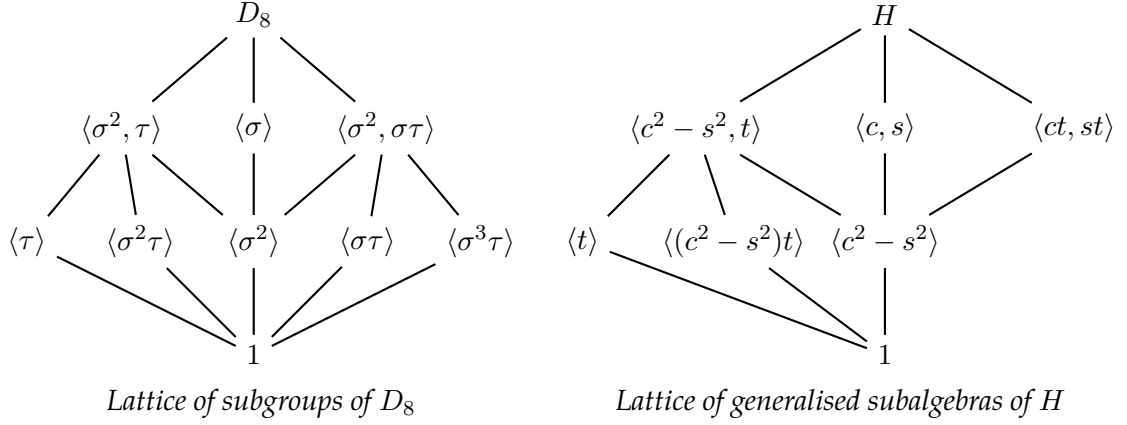
and thus $\Delta \otimes id \circ \Delta(c) = id \otimes \Delta \circ \Delta(c)$, similarly

$$\begin{aligned}\Delta \otimes id \circ \Delta(s) &= c \otimes c \otimes s - s \otimes s \otimes s + s \otimes c \otimes c + c \otimes s \otimes c \\ id \otimes \Delta \circ \Delta(s) &= c \otimes c \otimes s + c \otimes s \otimes c + s \otimes c \otimes c - s \otimes s \otimes s\end{aligned}$$

Thus $\Delta \otimes id \circ \Delta(s) = id \otimes \Delta \circ \Delta(s)$. The counit is given by $\epsilon(c) = 1$, $\epsilon(s) = 0$ and $\epsilon(t) = 1$. Finally, the antipode is defined by $S(c) = c$, $S(s) = -s$ and $S(t) = t$. Let us note that $\dim_{\mathbf{k}} H = 8$, the set

$$1, t, c, s, c^2 - s^2, tc, ts, t(c^2 - s^2)$$

is a linear basis. There are four group-like elements in H : $1, t, c^2 - s^2, t(c^2 - s^2)$. They form an abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The Hopf subalgebra H generated by c and s is called *circle Hopf algebra*. The name *trigonometric Hopf algebra* is also met in the literature. Furthermore, let us note that $\mathbf{k}[i] \otimes_{\mathbf{k}} H$ is isomorphic to the group Hopf algebra $\mathbf{k}[i][D_8]$, where D_8 is the dihedral group of order 8, i.e. the group of symmetries of a rectangle. For this note that $c - is$ is a group-like element of order 4 such that $t(c - is)t = (c - is)^3$. The claim follows since $\mathbf{k}[i] \otimes_{\mathbf{k}} H$ has 8 group-like elements and its dimension over $\mathbf{k}[i]$ is 8. In this identification t corresponds to a symmetry and $c - is$ is a rotation. The lattice of generalised subalgebras of H and the lattice of subgroups of D_8 are drawn below. We let τ and σ be the generators of D_8 , such that $\tau^2 = 1$, $\sigma^4 = 1$, $\tau\sigma\tau = \sigma^3$.



The notation $\langle \dots \rangle$ denotes the subgroup/left coideal subalgebra generated by the listed elements². For example the coideal subalgebra which is generated by $c^2 - s^2$ is spanned by c^2, s^2 and 1. The left coideal subalgebra $\langle ct, st \rangle$ also contains $c^2 = (ct)^2, s^2 = st(-st)$. Thus it contains $c^2 - s^2$. The lattice of generalised subalgebras of H is anti-isomorphic with the lattice of generalised quotients of its dual H^* .

The action of H on A is defined in table 4.1 below. This action makes A

	1	X^j	J	Z	Z^2	Z^3	Y
c	1	X^j	J	0	$-Z^2$	0	Y
s	0	0	0	$-Z$	0	Z^3	0
t	1	X^j	$-J$	Z	Z^2	Z^3	Y
c^2	1	X^j	J	0	Z^2	0	Y
s^2	0	0	0	Z	0	Z^3	0

Table 4.1: H -action on A

an H -module algebra. The Hopf algebra H is finite dimensional, hence its dual coacts on A . Moreover, the subalgebra of coinvariants of the H^* coaction is equal to the subalgebra of invariants of the H -action. We now show that A^H is the subalgebra generated by X . If $a = a' + a''J \in A^H$ is an invariant element, where both a', a'' belong to the subalgebra generated by X and Z , then we must have $a'' = 0$. Let $a = \sum_{i=0}^{15} f_i Z^i$, where each f_i is a polynomial

²The meet in the lattice of right coideal subalgebras of a Hopf algebra, which is a flat Mittag-Leffler module, is the set theoretic intersection. Thus we can define the left coideal subalgebra of H generated by some set of elements as the intersection of all left coideal subalgebras which contains these elements. This left coideal subalgebra will contain the elements by which it is generated. Also note that if the span of generating elements is a left coideal then the smallest subalgebra which contains it is a left coideal subalgebra.

in X . Hence:

$$0 = s \cdot a = \sum_{k=0}^7 (-1)^{k \bmod 2 + 1} f_{2k+1} Z^{2k+1}$$

In this way all the $f_{2k+1} = 0$ ($0 \leq k \leq 7$). Moreover,

$$a = c \cdot a = \sum_{k=0}^3 f_{4k} Z^{4k} - \sum_{k=0}^3 f_{4k+2} Z^{4k+2}$$

It follows that $f_{4k+2} = 0$ for $k = 0, 1, 2, 3$. Thus $A^H \subseteq B$. Conversely, every element of B belongs to A^H and hence $B = A^H = A^{co H^*}$. Furthermore, the extension A/B is a H^* -Galois. In order to show this we compute the following matrix. The columns are values of $can : A \otimes_B A \rightarrow \text{Hom}_{\mathbf{k}}(H, A)$ on the elements written in the first row.

	$1 \otimes_B 1$	$1 \otimes_B J$	$1 \otimes_B Z$	$1 \otimes_B JZ$	$1 \otimes_B Z^2$	$1 \otimes_B JZ^2$	$1 \otimes_B Z^3$	$1 \otimes_B JZ^3$
1_H	1	J	Z	JZ	Z^2	JZ^2	Z^3	JZ^3
t	1	$-J$	Z	$-JZ$	Z^2	$-JZ^2$	Z^3	$-JZ^3$
c	1	J	0	0	$-Z^2$	$-JZ^2$	0	0
s	0	0	$-Z$	$-JZ$	0	0	Z^3	JZ^3
c^2	1	J	0	0	Z^2	JZ^2	0	0
ct	1	$-J$	0	0	$-Z^2$	JZ^2	0	0
st	0	0	$-Z$	JZ	0	0	Z^3	$-JZ^3$
$c^2 t$	1	$-J$	0	0	Z^2	$-JZ^2$	0	0

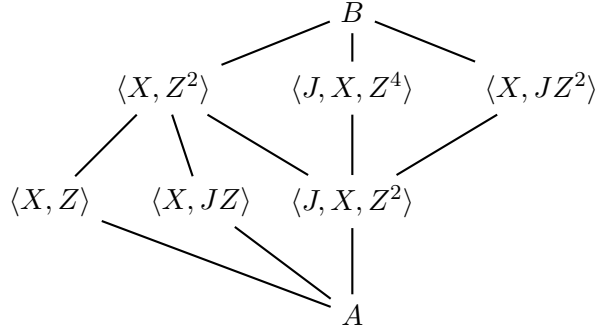
From this eight by eight matrix we get the following matrix (by normalising all the columns):

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

Since $\det M \neq 0$ (which we have verified using *Python*³) the map $can : A \otimes_B A \rightarrow \text{Hom}_{\mathbf{k}}(H, A)$ is injective. It is an epimorphism since $\dim_{\mathbf{k}} A \otimes_B A = \dim_{\mathbf{k}} \text{Hom}_{\mathbf{k}}(H, A) = 8 \cdot \dim_{\mathbf{k}} A < \infty$.

The image of $\phi : \text{Quot}_{gen}(H) \rightarrow \text{Sub}_{Alg}(A/B)$ is the following poset:

³We used the `numpy.linalg` module. See <http://docs.scipy.org/doc/numpy>



The image of $\text{Quot}_{\text{gen}}(H^*)$ in $\text{Sub}_{\text{Alg}}(A/B)$

Note that $A^{\langle c^2-s^2 \rangle} = \langle J, X, Z^2 \rangle$, $A^{\langle ct, st \rangle} = A^{\langle c^2-s^2, ct, st \rangle} = \langle X, JZ^2 \rangle$. We will later see that the morphism $\phi : \text{Quot}_{\text{gen}}(H^*) \rightarrow \text{Sub}_{\text{Alg}}(A/B)$ is a monomorphism for any finite H^* -Galois extension.

4.3 Closed Elements

The main aim of this section is to characterise the closed elements of the correspondence (4.5). First we provide two new important criteria for closedness of both: subalgebras of a comodule algebra and generalised quotients of the Hopf algebra. The criterion for generalised quotients is obtained using new methods based on both poset and Hopf algebraic methods. As a consequence of this we get that if the Hopf algebra is finite dimensional and the comodule algebra is a Hopf–Galois extension, then every generalised quotient is a closed element. Then we generalise the methods of Schauenburg: we redefine admissibility (cf. Definition 4.1.5 on page 76), and we show that admissible objects are closed elements of the Galois connection (4.5). We were able to slightly simplify the Schauenburg line of proof by proving a finer version of [Schauenburg, 1998, Prop. 3.5] (cf. Lemma 4.4.4 on page 92).

Theorem 4.3.1

Let A/B be an H -Galois extension such that A and H are flat Mittag-Leffler \mathbf{R} -modules, and let A be a faithfully flat left B -module. Then a subalgebra $S \in \text{Sub}_{\text{Alg}}(A/B)$ is a closed element of the Galois connection (4.5) if and only if the following canonical map is an isomorphism:

$$\text{can}_S : S \otimes_B A \longrightarrow A \square_{\psi(S)} H, \quad \text{can}_S(a \otimes b) = ab_{(1)} \otimes b_{(2)} \quad (4.8)$$

Proof: First let us note that the map can_S is well defined since it is a composition of $S \otimes_B A \rightarrow A^{\text{co}\psi(S)} \otimes_B A$, induced by the inclusion $S \subseteq A^{\text{co}\psi(S)}$, with $A^{\text{co}\psi(S)} \otimes_B A \rightarrow A \square_{\psi(S)} H$ which is an isomorphism by [Schneider, 1992b, Thm 1.4]. If S is closed then $S = A^{\text{co}\psi(S)}$, hence (4.8) is an isomorphism. Conversely, let us assume that can_S is an isomorphism. Then we have a commutative diagram:

$$\begin{array}{ccc}
S \otimes_B A & \xrightarrow{\text{can}_S} & A \square_{\psi(S)} H \\
\text{in} \nearrow & & \nearrow \text{can}_{A^{co\psi(S)}} \\
A^{co\psi(S)} \otimes_B A & &
\end{array}$$

where both can_S and $\text{can}_{A^{co\psi(S)}}$ are isomorphisms. Now, since A is a faithfully flat left B -module we must have $S = A^{co\psi(S)}$. Thus S is a closed element of (4.5). \square

Theorem 4.3.2

Let A be an H -comodule algebra over a ring \mathbf{R} with surjective canonical map and let A be a Q_1 -Galois and a Q_2 -Galois extension, where $Q_1, Q_2 \in \text{Quot}_{gen}(H)$. Furthermore, let us assume that one of the following assumptions holds:

1. Q_1 and Q_2 are flat as \mathbf{R} -modules and $\mathbf{R} \cong \mathbf{R}1_A$ as \mathbf{R} -modules, via the unit map $1_A : \mathbf{R} \rightarrow A$, $r \mapsto r1_A$ (hence A is a faithful \mathbf{R} -module); or
2. the unit map $1_A : \mathbf{R} \rightarrow A$ is a pure monomorphism (e.g. it has a left inverse).

Then the following implication holds:

$$A^{coQ_1} = A^{coQ_2} \Rightarrow Q_1 = Q_2$$

Proof: Let $B' = A^{coQ_1} = A^{coQ_2}$. The following diagram commutes:

$$\begin{array}{ccccc}
& & & A \otimes Q_1 & \\
& \nearrow \text{can}_{Q_1} & & \uparrow id \otimes \pi_1 & \\
A \otimes_{B'} A & \longleftarrow A \otimes_{A^{coH}} A & \xrightarrow{\text{can}} & A \otimes H & \\
& \searrow \text{can}_{Q_2} & & \downarrow id \otimes \pi_2 & \\
& & & A \otimes Q_2 &
\end{array}$$

The maps can_{Q_1} and can_{Q_2} are isomorphisms. Let $f := (\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1}) \circ (id \otimes \pi_2)$. By commutativity of the above diagram, $f \circ \text{can}$ and $(id \otimes \pi_1) \circ \text{can}$ are equal. Now, surjectivity of can yields the equality $(\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1}) \circ (id \otimes \pi_2) = (id \otimes \pi_1)$. We are going to construct $\pi : Q_1 \rightarrow Q_2$ such that $\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1} = id \otimes \pi$ and $\pi \circ \pi_2 = \pi_1$. Define $\pi : Q_2 \rightarrow Q_1$ by $\pi(\pi_2(h)) = \pi_1(h)$ for $h \in H$. If $\pi_2(h) = \pi_2(h')$ for $h, h' \in H$ then

$$\begin{aligned}
1_A \otimes \pi_1(h) &= (\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1})(1_A \otimes \pi_2(h)) \\
&= (\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1})(1_A \otimes \pi_2(h')) \\
&= 1_A \otimes \pi_1(h')
\end{aligned}$$

By flatness of Q_1 and the condition $\mathbf{R}1_A \cong \mathbf{R}$ or purity of the unit map we have $Q_1 = \mathbf{R}1_A \otimes Q_1 \subseteq A \otimes Q_1$. Now, it follows that $\pi_1(h) = \pi_1(h')$. This shows that $\pi : Q_2 \rightarrow Q_1$ is well defined. Furthermore, π is right H -linear and H -colinear:

$$\begin{aligned} \Delta_{Q_1}(\pi(\pi_2(h))) &= \Delta_{Q_1}(\pi_1(h)) \\ &= \pi_1 \otimes \pi_1 \circ \Delta_H(h) \\ &= (\pi \circ \pi_2)^{\otimes 2} \circ \Delta_H(h) \\ &= \pi \otimes \pi \Delta_{Q_2}(\pi_2(h)) \end{aligned}$$

Thus $Q_2 \succcurlyeq Q_1$. In the same way, using $(\text{can}_{Q_2} \circ \text{can}_{Q_1}^{-1}) \circ (\text{id} \otimes \pi_1)$ instead of f , we obtain that $Q_1 \succcurlyeq Q_2$. Now, by antisymmetry of \succcurlyeq , we get $Q_1 = Q_2$. \square

Before deriving a corollary from the above statement let us define an important class of generalised quotients. It depends on the comodule algebra and we are on the way to show that they are closed elements of our Galois correspondence. In some cases we will be able to show that this class is equal to the subset of closed quotients of the Hopf algebra.

Definition 4.3.3

Let A be an H -comodule algebra, where H is a Hopf algebra. Then for a generalised quotient $\pi : H \rightarrow Q$, $A/A^{\text{co}Q}$ is called *Q -Galois* if the following map is an isomorphism:

$$\text{can}_Q : A \otimes_{A^{\text{co}Q}} A \rightarrow A \otimes Q, \quad \text{can}(a \otimes_{A^{\text{co}Q}} b) = ab_{(0)} \otimes \pi(b_{(1)})$$

The following corollary explains why we rather tend to think that the above defines a class of quotients of H than subalgebras of A .

Corollary 4.3.4

Let A be an H -comodule algebra with epimorphic canonical map can_H such that the Galois connection (4.5) exists and let the unit map $1_A : \mathbf{R} \rightarrow A$ be a pure monomorphism. Then $Q \in \text{Quot}_{\text{gen}}(H)$ is a closed element of the Galois connection (4.5) if $A/A^{\text{co}Q}$ is Q -Galois.

This statement might be understood as a counterpart of Theorem 4.3.1 for generalised quotients.

Proof: Fix a coinvariant subalgebra $A^{\text{co}Q}$ for some $Q \in \text{Quot}_{\text{gen}}(H)$ then $\phi^{-1}(A^{\text{co}Q})$ is an upper-sublattice of $\text{Quot}_{\text{gen}}(H)$ (i.e. it is a subposet closed under finite suprema) which has a greatest element, namely $\tilde{Q} = \psi(A^{\text{co}Q})$. Moreover, \tilde{Q} is the only closed element belonging to $\phi^{-1}(A^{\text{co}Q})$. Both $Q \preccurlyeq \tilde{Q}$ and the assumption that $A/A^{\text{co}Q}$ is Q -Galois imply that $A/A^{\text{co}\tilde{Q}}$ is \tilde{Q} -Galois. To this end, we consider the commutative diagram:

$$\begin{array}{ccc}
A \otimes_B A & \xrightarrow{\text{can}_H} & A \otimes H \\
\downarrow & & \downarrow \\
A \otimes_{A^{co \tilde{Q}}} A & \xrightarrow{\text{can}_{\tilde{Q}}} & A \otimes \tilde{Q} \\
\downarrow = & & \downarrow \\
A \otimes_{A^{co Q}} A & \xrightarrow[\text{can}_Q]{\simeq} & A \otimes Q
\end{array}$$

From the lower commutative square we get that $\text{can}_{\tilde{Q}}$ is a monomorphism and from the upper commutative square we deduce that $\text{can}_{\tilde{Q}}$ is onto. Unless $\tilde{Q} = Q$ we get a contradiction with the previous proposition. \square

The above result applies also to the Galois correspondence (4.3), as it is the same as the Galois connection 4.5 for the left $L(A, H)$ -comodule algebra A (see Proposition 1.2.2(iii)). For a finite dimensional Hopf algebra H for every Q the extension $A/A^{co Q}$ is Q -Galois (see [Schauenburg and Schneider, 2005, Cor. 3.3]). Thus we get the following statement.

Proposition 4.3.5

Let H be a finite dimensional Hopf algebra over a field \mathbf{k} . Let A/B be an H -Galois extension. Then every $Q \in \text{Quot}_{\text{gen}}(H)$ is closed.

Consequently, the map ϕ of the Galois connection (4.5) is always a monomorphism, when the conditions of the above proposition are met.

4.4 Generalisation of Schauenburg's correspondence of admissible objects

In this section we generalise Theorem 4.1.7 (page 77). The line of proof is essentially the same as Schauenburg's, though many details had to be changed, and also some additional lemmas had to be proven. We make one assumption which did not appear in the original approach: in order to use Corollary 4.3.4 we assume that the unit map of the comodule algebra A is a pure monomorphism. The usage of Corollary 4.3.4 allows us to slightly simplify the proof. Also in our situation we do not assume, as Schauenburg did, that the coinvariants subalgebra is a central subalgebra, hence the single faithful flatness condition of A over the coinvariants has to be changed into faithful flatness as both a right and a left module.

In this section we will make heavy use of the opposite multiplication on both an H -comodule algebra A and the Hopf algebra H . The opposite algebra of A (or H) will be denoted by A^{op} (H^{op} respectively). Note that H^{op} is a bialgebra, with the same comultiplication as H . Furthermore, it is a Hopf

algebra if the antipode of H is bijective and $S_{H^{op}} = S_H^{-1}$, where S_H denotes the antipode of H . The right H^{op} -comodule structure of A^{op} is simply: $A^{op} \ni a \rightarrow a_{(0)} \otimes a_{(1)} \in A^{op} \otimes H^{op}$. Clearly, it is an algebra map, which makes A^{op} into an H^{op} -comodule algebra.

We let (ϕ^{op}, ψ^{op}) denote the Galois connection (4.5) for the H^{op} -comodule algebra A^{op} . Note that (ϕ^{op}, ψ^{op}) is a Galois correspondence if and only if (ϕ, ψ) is, by Lemma 4.4.4. We recall the following notation introduced in Proposition 4.1.6: for a generalised quotient $Q = H/I \in \text{Quot}_{gen}(H)$ we put $Q^{op} := H^{op}/S_H(I) \in \text{Quot}_{gen}(H^{op})$ and we also write $I^{op} := S_H(I)$. Now, we are ready to adapt Definition 4.1.5 to the more general context of H -extensions with noncommutative coinvariant subrings. We only amend the definition of admissible subalgebras of A , the coalgebra part is repeated.

Definition 4.4.1

1. Let C be an \mathbf{R} -coalgebra and let $C \rightarrow D$ be a coalgebra quotient. Then D is called **left (right) admissible** if it is \mathbf{R} -flat (hence faithfully flat) and C is left (right) faithfully coflat over D . For $I \in \text{cold}(C)$ we will say that it is **left (right) admissible** if and only if the quotient C/I shares this property.

Let A/B be an H -extension, where the Hopf algebra H has a bijective antipode.

1. Let S belong to $\text{Sub}_{Alg}(A/B)$. Then S is called **right admissible** if:
 - (a) A is right faithfully flat over S ,
 - (b) the composition:

$$\begin{aligned} \text{can}_S : A \otimes_S A &\rightarrow A \otimes_{A^{co\psi(S)}} A \xrightarrow{\text{can}_{\psi(S)}} A \otimes \psi(S) \\ a \otimes_S b &\mapsto ab_{(0)} \otimes b_{(1)}, \end{aligned}$$

is a bijection (it is well defined since $S \subseteq A^{co\psi(S)}$).

2. We call S **left admissible** if:
 - (a) A is left faithfully flat over S ,
 - (b) the composition:

$$\begin{aligned} \text{can}_S : A \otimes_S A &\rightarrow A \otimes_{A^{co\psi(S)}} A \xrightarrow{\text{can}_{\psi(S)}} A \otimes \psi(S) \\ a \otimes_S b &\mapsto ab_{(0)} \otimes b_{(1)}, \end{aligned}$$

is a bijection.

An element is called **admissible** if it is both left and right admissible.

Remark 4.4.2 If S is left or right admissible and A is faithfully flat over the base ring \mathbf{R} then $\psi(S)$ is flat as an \mathbf{R} -module.

Later, in Lemma 4.4.6, we will show that a subalgebra S is left (right) admissible if and only if $S^{op} \subseteq A^{op}$ is right (left) admissible. The same holds for

admissibility of generalised quotients if H is a flat Hopf algebra with a bijective antipode (see Proposition 4.1.6 on page 77).

The above notion of right and left admissibility for intermediate subalgebras of an extension A/B is equivalent to the original definition of Schauenburg when one considers A/B as a left $L(A, H)$ -Galois extension, and when we assume that $B = A^{co H}$ is equal to the ground ring \mathbf{R} , as is required in Schauenburg's approach. It is so since then the conditions (ii.b) and (iii.b) are automatically satisfied. The condition (ii.b) is satisfied by the way $L(A, H)$ is constructed. This is shown in Lemma 4.1.2 (page 75). The condition (iii.b) follows in the same way under the assumption that H has a bijective antipode, since then $L(A^{op}, H^{op}) \cong L(A, H)^{op}$ by the universal property of $L(A, H)$: if the antipode of H is bijective then A is right H -Galois if and only if A^{op} is right H^{op} -Galois (this is a special case of Lemma 4.4.6 on page 93). This also holds for $L(A, H)$ and $L(A, H)^{op}$, since $L(A, H)$ has a bijective antipode whenever H has. In this way A^{op}/\mathbf{R} is $L(A, H)^{op}$ - H^{op} -Galois. We have $L(A^{op}, H^{op}) \cong L(A, H)^{op}$, by the universal property of $L(A, H)$ (Proposition 4.1.3 on page 76).

We will need the following version of [Schneider, 1992b, Rem. 1.2].

Remark 4.4.3 Let A/B be an H -extension, such that both R -modules A and H are flat Mittag-Leffler. Let $S \in \text{Sub}_{Alg}(A/A^{co H})$. Then the following holds:

1. if $\text{can}_S : A \otimes_S A \rightarrow A \otimes \psi(S)$ is an isomorphism and A is right or left faithfully flat over S then S is a closed element of (4.5);
2. if the natural projection $A \otimes_S A \rightarrow A \otimes_{A^{co \psi(S)}} A$ is a bijection and A is right or left faithfully flat over S then $S = A^{co \psi(S)}$, i.e. S is a closed element of $\text{Sub}_{Alg}(A/B)$ in (4.5).

Proof: First let us prove (ii). For this let us consider the commutative diagram:

$$\begin{array}{ccccc}
 S & \subseteq & A & \rightrightarrows & A \otimes_S A \\
 \uparrow & & \parallel & & \downarrow \wr \\
 A^{co \psi(S)} & \subseteq & A & \rightrightarrows & A \otimes_{A^{co \psi(S)}} A
 \end{array}$$

The four maps $A \rightrightarrows A \otimes_S A$ and $A \rightrightarrows A \otimes_{A^{co \psi(S)}} A$ send $a \in A$ to $a \otimes 1_A$ or $1_A \otimes a$ in the respective tensor products. Since the diagram commutes and since the upper row is an equaliser (by faithfully flat descent), the dashed arrow exists, i.e. $A^{co \psi(S)} \subseteq S$. We get the equality since, $S \subseteq A^{co \psi(S)}$ holds by the Galois connection property.

The first claim follows from (ii) when applied to $S \subseteq A^{co \psi(S)}$, because the natural epimorphism $A \otimes_S A \rightarrow A \otimes_{A^{co \psi(S)}} A$ is an isomorphism whenever $\text{can}_S : A \otimes_S A \rightarrow A \otimes \psi(S)$ is an isomorphism:

$$\begin{array}{ccc}
A \otimes_S A & \xrightarrow[\cong]{\text{can}_S} & A \otimes \psi(S) \\
\downarrow & \nearrow \text{can}_{A^{co} \psi(S)} & \\
A \otimes_{A^{co} \psi(S)} A & &
\end{array}$$

□

The following lemma is a finer version of [Schauenburg, 1998, Prop. 3.5], since we are able to prove the equality $\psi^{op}(S^{op}) = (\psi(S))^{op}$ for any $S \in \text{Sub}_{Alg}(A/B)$:

Lemma 4.4.4

Let H be a Hopf algebra with a bijective antipode. Let A/B be an H -extension. Then the Galois connection (4.5) exists for the H^{op} -comodule algebra A^{op} if and only if it exists for the H -extension A/B . Furthermore, if this is the case

$$\phi^{op}(Q^{op}) = (\phi(Q))^{op} \text{ and } \psi^{op}(S^{op}) = (\psi(S))^{op}$$

where $S \in \text{Sub}_{Alg}(A/B)$ and $Q \in \text{Quot}_{gen}(H)$.

Proof: The first equation: $\phi^{op}(Q^{op}) = (\phi(Q))^{op}$ is proved in [Schauenburg, 1998, Prop. 3.5]. Both maps: $\text{Sub}_{Alg}(A/B) \ni S \mapsto S^{op} \in \text{Sub}_{Alg}(A^{op}/B^{op})$ and $\text{Quot}_{gen}(H) \ni Q \mapsto Q^{op} \in \text{Quot}_{gen}(H^{op})$ are isomorphisms of posets (see Proposition 4.1.6 or [Schauenburg, 1998, Prop. 3.5]). Thus ϕ reflects suprema if and only if ϕ^{op} reflects them. This shows that the Galois connection exists for A if and only if it exists for A^{op} due to Theorem 1.2.6). It remains to show that $\psi^{op}(S^{op}) = (\psi(S))^{op}$. First, let us observe that for any set \mathcal{O} of right ideal coideals we have $\bigwedge_{I \in \mathcal{O}} I^{op} = (\bigwedge_{I \in \mathcal{O}} I)^{op}$ in $\text{Id}_{gen}(H^{op})$:

$$\bigwedge_{I \in \mathcal{O}} I^{op} = \sum_{\substack{J \in \text{Id}_{gen}(H^{op}) \\ J \subseteq \bigcap_{I \in \mathcal{O}} I^{op}}} J = \sum_{\substack{J \in \text{Id}_{gen}(H) \\ J^{op} \subseteq (\bigcap_{I \in \mathcal{O}} I)^{op}}} J^{op} = \sum_{\substack{J \in \text{Id}_{gen}(H) \\ J \subseteq \bigcap_{I \in \mathcal{O}} I}} J^{op} = \left(\bigwedge_{I \in \mathcal{O}} I \right)^{op} \quad (4.9)$$

Let $\bar{\psi}(S) = \ker(H \rightarrow \psi(S))$. Now using formula (4.6) (on page 79) for ψ we get:

$$\begin{aligned}
\bar{\psi}(S) &= \bigwedge \left\{ I : S \subseteq A^{co H/I} \right\} \\
\bar{\psi}^{op}(S^{op}) &= \bigwedge \left\{ I^{op} : S^{op} \subseteq (A^{op})^{co H^{op}/I^{op}} \right\}
\end{aligned} \quad (4.10)$$

Since $(A^{co H/I})^{op} = (A^{op})^{co H^{op}/I^{op}}$ and $\text{Id}_{gen}(H) \ni I \mapsto I^{op} := S_H(I) \in \text{Id}_{gen}(H^{op})$ is a bijection, the two sets in (4.10) are in a bijective correspondence via $I \mapsto I^{op}$. Now, the formula $\psi^{op}(S^{op}) = (\psi(S))^{op}$ follows from equation (4.9). □

Another way of proving the equation $\psi^{op}(S^{op}) = (\psi(S))^{op}$ is to show that

the map $\chi : \text{Sub}_{\text{Alg}}(A^{op}/B^{op}) \rightarrow \text{Quot}_{\text{gen}}(H^{op})$ given by $\chi(S) := (\psi(S^{op}))^{op}$, where $S \in \text{Sub}_{\text{Alg}}(A^{op}/B^{op})$, is an adjoint for ϕ^{op} and conclude the argument referring to uniqueness of Galois connections.

Proposition 4.4.5

Let H be a Hopf algebra with a bijective antipode. Let A be an H -extension of B such that A is H -Galois, A is a faithfully flat Mittag-Leffler module, H is a flat Mittag-Leffler \mathbf{R} -module, A is faithfully flat as both a left and right B -module, and also faithfully flat as an \mathbf{R} -module and finally let us assume that $1_A : \mathbf{R} \rightarrow A$ is pure. Then:

1. if $S \in \text{Sub}(A/B)$ is right admissible then so is $\psi(S)$ and $\phi\psi(S) = S$,
2. if $Q \in \text{Quot}_{\text{gen}}(H)$ is left admissible then so is $\phi(Q) := A^{co}Q$ and moreover $\psi\phi(Q) = Q$.

Proof: We first prove (ii): $\phi(Q)$ is left admissible by applying [Schneider, 1992b, Thm 1.4] to A . The equality $\psi\phi(Q) = Q^{op}$ follows from Corollary 4.3.4. Suppose, that $S \in \text{Sub}(A/B)$ is a right admissible subalgebra. Then by Remark 4.4.3(i) $S = \phi\psi(S)$ and A/S is Q -Galois. The proof that $\psi(S)$ is a faithfully flat \mathbf{R} -module and H is faithfully coflat as a right $\psi(S)$ -comodule remains the same as in [Schauenburg, 1998, Prop. 3.4]. For the sake of completeness let us recall these arguments. First of all $\psi(S)$ is a flat \mathbf{R} -module, since we have an isomorphism $A \otimes_S A \cong A \otimes \psi(S)$ (by right admissibility of S) and A is faithfully flat as a right S -module and also as an \mathbf{R} -module. For any left $\psi(S)$ -comodule V we have a chain of isomorphisms:

$$A \otimes (H \square_{\psi(S)} V) \cong (A \otimes H) \square_{\psi(S)} V \cong (A \otimes_B A) \square_{\psi(S)} H \cong A \otimes_B (A \square_{\psi(S)} V)$$

Now, H is faithfully coflat as a right $\psi(S)$ -comodule, since A is faithfully flat as an \mathbf{R} -module, faithfully flat as a right B -module and faithfully coflat as a right $\psi(S)$ -comodule by [Schneider, 1992b, Rem. 1.2(2)]. \square

Now we will show that the property of being a left/right admissible subalgebra is symmetric with respect to taking the opposite algebra.

Lemma 4.4.6

Let H be a Hopf algebra with a bijective antipode and let A be an H -comodule algebra. Then $S \in \text{Sub}_{\text{Alg}}(A)$ is left (right) admissible if and only if $S^{op} \subseteq A^{op}$ is right (left) admissible.

Proof: Note that A is left (right) faithfully flat over S if and only if A^{op} is right (left) faithfully flat over S^{op} . It remains to show that $\text{can}_S : A \otimes_S A \rightarrow A \otimes \psi(S)$ is an isomorphism if and only if $\text{can}_{S^{op}} : A^{op} \otimes_{S^{op}} A^{op} \rightarrow A^{op} \otimes \psi^{op}(S^{op})$ is.

Let (ϕ^{op}, ψ^{op}) be the Galois correspondence (4.5) for the H^{op} -comodule algebra A^{op} . Let us consider $Q = H/I \in \text{Quot}_{\text{gen}}(H)$ with $\pi : H \rightarrow Q$ the natural projection. Then $Q^{op} = H^{op}/S_H(I) \in \text{Quot}_{\text{gen}}(H^{op})$, or equivalently Q^{op} is a left generalised quotient of H (i.e. quotient by a left ideal coideal).

We let $\pi^{op} : H^{op} \rightarrow Q^{op}$ denote the natural projection map. The antipode S_H of H induces an isomorphism $\overline{S_H} : Q \rightarrow Q^{op}$ which is given by $\overline{S_H}(\pi(h)) := \pi^{op}(S_H(h))$, $h \in H$. Its inverse is given by $Q^{op} \ni \pi^{op}(h) \mapsto \pi(S_H^{-1}(h)) \in Q$. Let us show that $\overline{S_H} : Q \rightarrow Q^{op}$ is an H -module isomorphism (over the ring homomorphism: $S_H : H \rightarrow H^{op}$):

$$\begin{aligned} \overline{S_H}(\pi(h)k) &= \overline{S_H}(\pi(hk)) \\ &= \pi^{op}(S_H(hk)) \\ &= \pi^{op}(S_H(k)S_H(h)) \\ &= S_H(k)\pi^{op}(S_H(h)) \\ &= S_H(k)\overline{S_H}(\pi(h)) \end{aligned}$$

for all $h, k \in H$. Finally let us note that $\psi^{op}(S^{op}) = (\psi(S))^{op}$ by Lemma 4.4.4. We let $Q = \psi(S)$, $\pi : H \rightarrow \psi(S)$.

We have the following commutative diagram:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\text{can}_S} & A \otimes \psi(S) \\ \tau \downarrow & & \downarrow \alpha \\ A^{op} \otimes_{S^{op}} A^{op} & \xrightarrow{\text{can}_{S^{op}}} & A^{op} \otimes \psi^{op}(S^{op}) \end{array} \quad (4.11)$$

where:

$$\begin{aligned} \text{can}_S : A \otimes_S A &\rightarrow A \otimes \psi(S), & \text{can}_S(a \otimes b) &:= ab_{(0)} \otimes b_{(1)} \\ \text{can}_{S^{op}} : A^{op} \otimes_{S^{op}} A^{op} &\rightarrow A^{op} \otimes \psi^{op}(S^{op}), & \text{can}_{S^{op}}(a \otimes b) &:= b_{(0)}a \otimes b_{(1)} \\ \tau : A \otimes_S A &\rightarrow A^{op} \otimes_{S^{op}} A^{op}, & \tau(a \otimes b) &= b \otimes a \\ \alpha : A \otimes \psi(S) &\rightarrow A^{op} \otimes \psi^{op}(S^{op}), & \alpha(a \otimes \pi(h)) &:= a_{(0)} \otimes a_{(1)}\overline{S_H}(\pi(h)) \end{aligned}$$

Note that we use the convention that concatenation ab for $a, b \in A$ always denotes the multiplication in A . The map τ is a well defined isomorphism. Also α is well defined since $\psi^{op}(S^{op}) = (\psi(S))^{op}$ is a left H -module. It is an isomorphism with inverse:

$$\begin{aligned} \alpha^{-1} : A^{op} \otimes \psi^{op}(S^{op}) &\rightarrow A \otimes \psi(S) \\ \alpha^{-1}(a \otimes \pi^{op}(h)) &:= a_{(0)} \otimes \overline{S_H}^{-1}(\pi^{op}(h))a_{(1)} \end{aligned}$$

Indeed α^{-1} is an inverse of α :

$$\begin{aligned} \alpha^{-1} \circ \alpha(a \otimes \pi(h)) &= \alpha^{-1}(a_{(0)} \otimes a_{(1)}\pi^{op}(S_H(h))) \\ &= \alpha^{-1}(a_{(0)} \otimes \pi^{op}(a_{(1)}S_H(h))) \\ &= a_{(0)} \otimes \pi(S_H^{-1}(a_{(1)}S_H(h)))a_{(1)} \\ &= a_{(0)} \otimes \pi(hS_H^{-1}(a_{(1)})a_{(1)}) \\ &= a \otimes \pi(h) \end{aligned}$$

The other equality $\alpha \circ \alpha^{-1} = id$ follows in a similar way. Now the claim readily follows from the above commutative diagram, since both τ and α are isomorphisms. \square

The following theorem is a generalisation of Schauenburg's Theorem 4.1.7.

Theorem 4.4.7

Let H be a Hopf algebra with a bijective antipode. Let A/B be an H -extension such that A/B is H -Galois and A is faithfully flat as both a left and right B -module. Furthermore, suppose that A is a faithfully flat Mittag-Leffler \mathbf{R} -module and H is a flat Mittag-Leffler module. Finally, let us assume that $1_A : \mathbf{R} \rightarrow A$ is pure. Then the Galois connection (4.5) gives rise to a bijection between (left, right) admissible objects, hence (left, right) admissible objects are closed.

Note that here we consider a right H -comodule algebras, while Schauenburg considers a left $L(A, H)$ -comodule algebra structures. We make one additional assumption, that the unit map $1_A : \mathbf{R} \rightarrow A$ is a pure monomorphism, since our proof makes use of Corollary 4.3.4. The Lemma 4.4.4 is stronger than the corresponding part of [Schauenburg, 1998, Prop. 3.5] which simplifies the first part of the following proof. Finally, (left, right) admissible generalised quotients of $L(A, H)$ classify (left, right) admissible H -comodule algebras, while (left, right) admissible quotients of H classify (left, right) admissible subalgebras of A .

Proof of Theorem 4.4.7: We let (ϕ^{op}, ψ^{op}) denote the Galois connection (4.5) for A^{op} a right H^{op} -comodule algebra instead of A – an H -comodule algebra.

Let $S \subseteq A$ be left admissible. Then $S^{op} \subseteq A^{op}$ is right admissible by Lemma 4.4.6. Hence $\psi^{op}(S^{op})$ is right admissible and $\phi^{op}\psi^{op}(S^{op}) = S^{op}$, by Proposition 4.4.5(1). It follows that $\phi\psi(S) = S$, by Lemma 4.4.4. Using [Schauenburg, 1998, Prop. 3.5], which shows that $I \in \text{Id}_{gen}(H)$ is left (right) admissible if $I^{op} \in \text{Id}_{gen}(H^{op})$ is right (left) admissible, we conclude that $\psi(S) = (\psi^{op}(S^{op}))^{op}$ is left admissible.

Let I be right admissible. Then I^{op} is left admissible, and so is the subalgebra $\phi^{op}(H^{op}/I^{op}) = \phi(H/I)^{op}$. Thus $\phi(H/I)$ is right admissible, and $H^{op}/I^{op} = \psi^{op}\phi^{op}(H^{op}/I^{op}) = (\psi\phi(H/I))^{op}$. Hence $H/I = \psi\phi(H/I)$, by the result of [Schauenburg, 1998, Prop. 3.5]. \square

Remark 4.4.8 *If we consider a faithfully flat H -Galois extension A/\mathbf{R} , then the above result applied to the left $L(A, H)$ -comodule algebra A reduces to Theorem 4.1.7 (on page 77). We already pointed out that in this situation the two extra conditions (ii.b) and (iii.b) in our definition of admissibility (Definition 4.4.1 on page 90) naturally follow in the Schauenburg context (see the paragraphs after Definition 4.4.1). The missing step is that in Schauenburg's Theorem 4.1.7 (right, left) admissible generalised quotients of $L(A, H)$ correspond to (right, left) admissible H -subcomodule*

algebras rather than just (right, left) admissible subalgebras of A . But right admissible subalgebras in the sense of Definition 4.4.1 for the left $L(A, H)$ -comodule algebra⁴ A are H -subcomodules by Remark 4.4.3 (on page 91) as every subalgebra of coinvariants for quotients of $L(A, H)$ is always an H -subcomodule, since A is an $L(A, H)$ - H -bicomodule. In a similar way left admissible subalgebras in the sense of Definition 4.4.1 (on page 90) are H -subcomodules.

4.5 Galois correspondence for Galois coextensions

In this section we describe the Galois theory for Galois coextensions. We begin with some basic definitions. The main theorem of this section is Theorem 4.5.5 (on page 97) which is a dual version of Theorem 4.3.2. We will need this result in the following section for a new proof of the Takeuchi correspondence, i.e. a bijection between generalised quotients and generalised Hopf subalgebras of a finite dimensional Hopf algebra (Theorem 4.6.1 on page 99). For the following two definitions we refer to Schneider [1990].

Definition 4.5.1

Let C be a coalgebra and H a Hopf algebra, both over a ring \mathbf{R} . We call C a left H -module coalgebra if it is a left H -module such that the H -action $H \otimes C \rightarrow C$ is a coalgebra map:

$$\Delta_C(h \cdot c) = \Delta_H(h)\Delta_C(c), \quad \epsilon_C(h \cdot c) = \epsilon_H(h)\epsilon_C(c).$$

Let $C^H := C/H^+C$ be the *invariant coalgebra*. Furthermore, we call $C \rightarrow C^H$ an H -coextension.

Definition 4.5.2

1. An H -module coalgebra C is called an H -Galois coextension if the canonical map

$$\text{can}_H : H \otimes C \rightarrow C \square_{C^H} C, \quad h \otimes c \mapsto hc_{(1)} \otimes c_{(2)}$$

is a bijection, where C is considered as a left and right C^H -comodule in a standard way.

2. More generally, if $K \in \text{cold}_l(H)$ is a left coideal then K^+C is a coideal (at least, when the base ring \mathbf{R} is a field) and an H -submodule of C . The coextension $C \rightarrow C^K = C/K^+C$ is called Galois if the canonical map

$$\text{can}_K : K \otimes C \rightarrow C \square_{C^K} C, \quad k \otimes c \mapsto kc_{(1)} \otimes c_{(2)} \quad (4.13)$$

is a bijection.

⁴Note that though Definition 4.4.1 (on page 90) is written for right comodule algebras we are using it here for the left $L(A, H)$ -comodule algebra A . For left comodule algebras their canonical map should be used in (ii.b) and (iii.b).

A basic example of an H -module coalgebra is H itself. Then $H^H = \mathbf{R}$ and $H \square_{H^H} H = H \otimes H$. The inverse of the canonical map is given by $\text{can}_H^{-1}(k \otimes h) = kS(h_{(1)}) \otimes h_{(2)}$.

Definition 4.5.3

Let C be an H -module coalgebra. We let $\text{Quot}(C) = \{C/I : I \text{ a coideal of } C\}$ with order relation $C/I_1 \succcurlyeq C/I_2 \Leftrightarrow I_1 \subseteq I_2$. It is a complete lattice. We let $\text{Quot}(C/C^H)$ denote the subset $\{Q \in \text{Quot}(C) : C \succcurlyeq Q \succcurlyeq C^H\}$ in $\text{Quot}(C)$.

Proposition 4.5.4

Let C be an H -module coalgebra over a field \mathbf{k} . Then there exists a Galois connection:

$$\begin{array}{ccc} \text{Quot}(C/C^H) & \xleftrightarrow{\quad} & \text{cold}_l(H) \\ C/(I + \mathbf{k}1_H)^+ C & \longleftarrow I & \end{array} \quad (4.14)$$

Proof: The supremum in $\text{cold}_l(H)$ is given by the sum of \mathbf{k} -subspaces. Thus the lattice of left coideals is complete. Furthermore, if I is a left coideal then $I + \mathbf{k}1_H$ is also a left coideal. Thus $(I + \mathbf{k}1_H)^+$ is a coideal of H . As a consequence $(I + \mathbf{k}1_H)^+ C$ is a coideal of C . It is enough to show that the map $\text{cold}_l(H) \ni I \mapsto I^+ C \in \text{cold}(C)$ preserves all suprema when we restrict to left coideals which contain 1_H . Let $I_\alpha \in \text{cold}(H)$ ($\alpha \in \Lambda$) be such that $1_H \in I_\alpha$ for all $\alpha \in \Lambda$. Then $(\sum_\alpha I_\alpha)^+ = \sum_\alpha (I_\alpha^+)$. The non trivial inclusion is $(\sum_\alpha I_\alpha)^+ \subseteq \sum_\alpha (I_\alpha^+)$. Let $k = \sum_\alpha k_\alpha \in (\sum_\alpha I_\alpha)^+$, i.e. $k_\alpha \in I_\alpha$ for all $\alpha \in \Lambda$ and $\sum_\alpha k_\alpha \in \ker \epsilon$. Then $\sum_\alpha k_\alpha = \sum_\alpha (k_\alpha - \epsilon(k_\alpha)1_H) + \sum_\alpha (\epsilon(k_\alpha)1_H) = \sum_\alpha (k_\alpha - \epsilon(k_\alpha)1_H)$. Each $k_\alpha - \epsilon(k_\alpha)1_H \in I_\alpha^+$ and hence $k \in \sum_\alpha I_\alpha^+$. Now the proposition follows easily. \square

Theorem 4.5.5

Let C be an H -module coalgebra over a field \mathbf{k} such that the canonical map can_H is injective. Let K_1, K_2 be two left coideals of H such that both can_{K_1} and can_{K_2} are bijections. Then $K_1 = K_2$ whenever $C^{K_1} = C^{K_2}$.

Proof: We have the following commutative diagram:

$$\begin{array}{ccccc} & & K_1 \otimes C & & \\ & \swarrow \text{can}_{K_1} & \downarrow i_1 \otimes \text{id} & \searrow & \\ C \square_{C^{K_1}} C & \xrightarrow{\quad} & C \square_{C^H} C & \xleftarrow{\text{can}} & H \otimes C \\ & \nwarrow \text{can}_{K_2} & \uparrow i_2 \otimes \text{id} & \swarrow & \\ & & K_2 \otimes C & & \end{array}$$

It follows that $i_2 \otimes id \circ (can_{K_2} \circ can_{K_1}^{-1}) = i_1 \otimes id$. Thus $K_1 \subseteq K_2$, since we are over a field; similarly $K_2 \subseteq K_1$. \square

Corollary 4.5.6

Let C be an H -coextension such that the canonical map can_H is injective. Then a left coideal K of H with $1_H \in K$ is a closed element of the Galois connection (4.14) if $C \rightarrow C^K$ is K -Galois.

Proof: Let C be a K -Galois coextension, for some left coideal K of H such that $1_H \in K$ and let \tilde{K} be the smallest closed left coideal such that $K \subseteq \tilde{K}$. Then we have the commutative diagram:

$$\begin{array}{ccc}
 H \otimes C & \xrightarrow{can_H} & C \square_{C^H} C \\
 \uparrow \wr & & \uparrow \wr \\
 \tilde{K} \otimes C & \xrightarrow{can_{\tilde{K}}} & C \square_{C^{\tilde{K}}} C \\
 \uparrow \wr & & \uparrow \parallel \\
 K \otimes C & \xrightarrow[\simeq]{can_K} & C \square_{C^K} C
 \end{array}$$

From the lower commutative square it follows that $can_{\tilde{K}}$ is onto, and from the upper one that it is a monomorphism. The result follows now from the previous theorem. \square

4.6 Takeuchi correspondence

We show a new simple proof of the Takeuchi correspondence between left coideal subalgebras and right H -module coalgebra quotients of a finite dimensional Hopf algebra. We also show that for an arbitrary Hopf algebra H , a generalised quotient Q is closed if and only if $H^{coQ} \subseteq H$ is Q -Galois. Similarly, for a left coideal subalgebras it is closed if and only if $H \rightarrow H^K$ is a K -Galois coextension.

Theorem 4.6.1

Let H be a bialgebra which is flat as an R -module and let us assume that the antipode of H is bijective. Then:

$$\begin{aligned}
 \left\{ K \subseteq H : K \text{ - left coideal subalgebra} \right\} & \xrightleftharpoons[\phi]{\psi} \left\{ H/I : I \text{ - right ideal coideal} \right\} \\
 =: \text{Sub}_{gen}(H) & \qquad \qquad \qquad =: \text{Quot}_{gen}(H)
 \end{aligned} \tag{4.15}$$

where $\phi(Q) = H^{\text{co}Q}$, $\psi(K) = H/K^+H$ is a Galois connection which is a restriction of the Galois connection (4.5). Moreover, if H is a Hopf algebra it restricts to normal Hopf subalgebras and conormal Hopf quotients, and the following holds:

- (i) $K \in \text{Sub}_{\text{gen}}(H)$ such that H is (left, right) faithfully flat over K , is a closed element of the above Galois connection,
- (ii) $Q \in \text{Quot}_{\text{gen}}(H)$ such that H is (left, right) faithfully coflat over Q and Q is flat as an R -module is a closed element of the Galois connection (4.4),
- (iii) if R is a field and H is finite dimensional, then ϕ and ψ are inverse bijections. The Galois correspondence restricts to a bijection between elements satisfying (i) and (ii).

The points (i) and (ii) follow from Theorem 4.4.7 (due to Schauenburg, see also [Schauenburg, 1998, Thm 3.10]), while point (iii) follows from [Skryabin, 2007, Thm 6.1], where it is shown that if H is finite dimensional then it is free over each of its right (or left) coideal subalgebras (see [Skryabin, 2007, Thm 6.6] and also [Schauenburg and Schneider, 2005, Cor. 3.3]). This theorem has a long history. The study of this correspondence, with Hopf algebraic method, goes back to Takeuchi [1972, 1979]. Then Masuoka proved (i) and (ii) for Hopf algebras over a field (with bijective antipode). When the base ring is a field, Schneider [1993, Thm 1.4] proved that this bijection restricts to normal Hopf subalgebras and normal Hopf algebra quotients. For Hopf algebras over more general rings it was shown by Schauenburg [1998, Thm 3.10]. We can present a new simple proof of 4.6.1(iii), which avoids the Skryabin result.

Proof of Theorem 4.6.1(iii): Whenever H is finite dimensional, for every Q the extension $H^{\text{co}Q} \subseteq H$ is Q -Galois by [Schauenburg and Schneider, 2005, Cor. 3.3]. Using Theorem 4.3.2 we get that the map ϕ is a monomorphism. To show that it is an isomorphism it is enough to prove that ψ is a monomorphism. We now want to consider H^* . To distinguish ϕ and ψ for H and H^* we will write ϕ_H and ψ_H considering (4.4) for H and ϕ_{H^*} and ψ_{H^*} considering H^* . It turns out that $(\psi_H(K))^* = \phi_{H^*}(K^*)$. Now we show that $\text{can}_{K^*} = (\text{can}_K)^* : H^* \otimes_{\text{co}K^*} H^* \rightarrow K^* \otimes H^*$ under some natural identifications (note that we consider H^* as a left K^* -comodule algebra). First let us observe that ${}^{\text{co}K^*}H^* = (H/K^+H)^*$:

$$\begin{aligned} {}^{\text{co}K^*}H^* &= \{f \in H^* : f_{(1)}|_K \otimes f_{(2)} = \epsilon|_K \otimes f\} \\ &= \{f \in H^* : \forall_{h \in H, k \in K} f_{(1)}(k)f_{(2)}(h) = \epsilon(k)f(h)\} \\ &= \{f \in H^* : f|_{K^+H} = 0\} \\ &= (H/K^+H)^* \end{aligned}$$

Now, since $H \square_{H^K} H$ is defined by the kernel diagram of finite dimensional spaces:

$$H \square_{H^K} H \longrightarrow H \otimes H \rightrightarrows H \otimes (H/K^+H) \otimes H$$

we get the cokernel diagram:

$$H^* \otimes {}^{co K^*} H^* \otimes H^* \rightrightarrows H^* \otimes H^* \longrightarrow (H \square_{H^K} H)^*$$

But the above exact sequence defines $H^* \otimes_{{}^{co K^*} H^*} H^*$. Now, we have a commutative diagram:

$$\begin{array}{ccc} (H \square_{H^K} H)^* & \xrightarrow{(can_K)^*} & K^* \otimes H^* \\ \uparrow \wr & \nearrow can_{K^*} & \\ H^* \otimes_{{}^{co K^*} H^*} H^* & & \end{array}$$

since $(can_K)^*(f \otimes g)(k \otimes h) = f \otimes g \circ can_K(k \otimes h) = f(kh_{(1)})g(h_{(2)}) = f_{(1)}(k)f_{(2)}(h_{(1)})g(h_{(2)}) = (f_{(1)} \otimes f_{(2)} * g)(k \otimes h) = can_{K^*}(f \otimes g)(k \otimes h)$.

The canonical map can_{K^*} is an isomorphism, since H^* is finite dimensional, and hence can_K is a bijection for every left coideal subalgebra K of H . Now the result follows from Theorem 4.5.5 and Proposition 1.2.2(iv). \square

Theorem 4.6.2

Let H be a Hopf algebra such that H is a flat \mathbf{R} -module. Then:

1. $Q \in \text{Quot}_{gen}(H)$ is a closed element of the Galois connection (4.4) if and only if H/H^{coQ} is a Q -Galois extension,
2. $K \in \text{Sub}_{gen}(H)$ is a closed element of the Galois connection (4.4) if and only if $H \rightarrow H^K$ is a K -Galois coextension (Definition 4.5.2(ii)), i.e. the map:

$$K \otimes H \rightarrow H \square_{H/K+H} H, \quad k \otimes h \mapsto kh_{(1)} \otimes h_{(2)}$$

is an isomorphism.

Note that we do not assume that the antipode of H is bijective as it is done in Theorem 4.6.1. The flatness of H is only needed to show that if K is closed then $H \rightarrow H^K$ is a K -Galois coextension.

Proof: For the first part, it is enough to show that if Q is closed then $H^{coQ} \subseteq H$ is Q -Galois (see Corollary 4.3.4). If Q is closed then $Q = H/(H^{coQ})^+H$. One can show that for any $K \in \text{Sub}_{gen}(H)$ the following map is an isomorphism:

$$H \otimes_K H \rightarrow H \otimes H/K^+H, \quad h \otimes_K h' \mapsto hh'_{(1)} \otimes \overline{h'_{(2)}} \quad (4.16)$$

Its inverse is given by $H \otimes H/K^+H \ni h \otimes \overline{h'} \mapsto hS(h'_{(1)}) \otimes_K h'_{(2)} \in H \otimes_K H$ which is well defined since K is a left coideal. Plugging $K = H^{coQ}$ into equation (4.16) we observe that this map is the canonical map (1.5) associated to Q .

Now, if $H \rightarrow H/K^+H$ is a K -Galois coextension then it follows from Theorem 4.5.5, using the same argument as in Corollary 4.5.6, that K is a closed element. Now, let us assume that K is closed. Then $K = H^{\text{co}Q}$ for $Q = H/K^+H$. By [Schneider, 1992b, Thm 1.4(1)] we have an isomorphism:

$$H^{\text{co}Q} \otimes H \longrightarrow H \square_Q H \quad k \otimes h \longmapsto kh_{(1)} \otimes h_{(2)} \quad (4.17)$$

with inverse $H \square_Q H \ni k \otimes h \longmapsto kS(h_{(1)}) \otimes h_{(2)} \in H^{\text{co}Q} \otimes H$. The above map is the canonical map (4.13) since $K = H^{\text{co}Q}$ and $Q = H/K^+H$. \square

Let us note that the statement (i) in the above result generalises [Schauenburg and Schneider, 2005, Corollary 3.3(6)]. Now we get an answer to the question when the bijection (4.4) holds without extra assumptions.

Corollary 4.6.3

The bijective correspondence (4.4) holds without flatness/coflatness assumptions if and only if for every $Q \in \text{Quot}_{\text{gen}}(H)$ $H/H^{\text{co}Q}$ is a Q -Galois extension and for every $K \in \text{Sub}_{\text{gen}}(H)$ H/H^K is a K -Galois coextension.

For Hopf algebras which are commutative (as algebras) or whose coradical is cocommutative, the correspondence (4.4) restricts to a bijection between normal Hopf subalgebras and normal quotients ([Montgomery, 1993, Thm 3.4.6]).

Corollary 4.6.4

Let H be a finite dimensional Hopf algebra and K be its left coideal subalgebra. Then H/H^K is K -Galois coextension.

Proof: Combine Theorem 4.6.1(iii) and Theorem 4.6.2(ii). \square

Example 4.6.5 Let \mathfrak{g} be a finite dimensional Lie algebra and let us consider the Hopf algebra $\mathcal{U}(\mathfrak{g})$ then the Takeuchi correspondence (4.4) is a bijection.

Proof: Takeuchi showed that (4.4) gives a bijective correspondence between coideal subalgebras of a commutative Hopf algebra H and generalised quotients over which H is faithfully coflat (see [Takeuchi, 1979, Thm. 4]). By Proposition 3.2.19 and Theorem 3.2.20 we get that every generalised quotient of $\mathcal{U}(\mathfrak{g})$ is of the form $\mathcal{U}(\mathfrak{g})/K^+\mathcal{U}(\mathfrak{g})$ for some coideal subalgebra K . \square

Moreover, by Proposition 3.2.19 and Theorem 3.2.20 we get that every coideal subalgebra of $\mathcal{U}(\mathfrak{g})$ is of the form $\mathcal{U}(\mathfrak{h})$ where \mathfrak{h} is a Lie subalgebra of \mathfrak{g} , and every generalised quotient is of the form $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{h})^+\mathcal{U}(\mathfrak{g})$.

Theorem 4.6.2 and Theorem 4.2.4 give the following

Corollary 4.6.6

Let A/B be an H -extension over a field k , where H is a Hopf algebra. Then every closed generalised quotient $Q \in \text{Quot}_{\text{gen}}(H)$ is of the form H/K^+H by

Theorem 4.2.4. *Thus Q cannot be closed in (4.5) if Q is not closed in the Galois connection (4.4), i.e. when $\text{can}_Q : H \otimes_{H^{\text{co}} Q} H \rightarrow H \otimes Q$ is not bijective.*

We can show what is actually lacking to get the converse of Corollary 4.3.4.

Theorem 4.6.7

Let A/B be an H -Galois extension over a field \mathbf{k} . Then the following two conditions are equivalent:

1. *the canonical map $\text{can}_Q : A \otimes_{A^{\text{co}} Q} A \rightarrow A \otimes Q$ is a bijection, and*
2. *(a) $Q \in \text{Quot}_{\text{gen}}(H)$ is closed in the Galois connection (4.5), and
(b) the map $\delta_A \otimes \delta_A : A \otimes_{A^{\text{co}} Q} A \rightarrow (A \otimes H) \otimes_{A \otimes H^{\text{co}} Q} (A \otimes H)$ is an injection.*

The map $\delta_A \otimes \delta_A$ is well defined since δ_A is a morphism of Q -comodules. However the splitting of δ_A given by the counit ϵ_H , does not define a splitting of $\delta_A \otimes \delta_A$ in general.

Proof: First let us assume that Q is a closed element. Then by theorem 4.2.4 $Q = H/K^+H$ for some $K \in \text{Sub}_{\text{gen}}(H)$ and thus Q is closed in (4.4). Hence, by Theorem 4.6.2(ii) the map $\text{can}_Q : H \otimes_{H^{\text{co}} Q} H \rightarrow H \otimes Q$ is a bijection. Now let us consider the following commutative diagram:

$$\begin{array}{ccc}
 A \otimes_{A^{\text{co}} Q} A & \xrightarrow{\delta_A \otimes \delta_A} & (A \otimes H) \otimes_{A \otimes H^{\text{co}} Q} (A \otimes H) \xrightarrow{\cong} A \otimes (H \otimes_{H^{\text{co}} Q} H) \\
 \text{can}_Q \downarrow & & \downarrow \text{id}_A \otimes \text{can}_Q \\
 A \otimes Q & \xrightarrow{\delta_A \otimes \text{id}_Q} & A \otimes H \otimes Q
 \end{array}$$

where δ_A is the H -comodule structure map. Now, since $\delta_A \otimes \delta_A$ is an injection and the right vertical map is a bijection we conclude that the left vertical homomorphism is a monomorphism, and it is onto since $A/A^{\text{co}} H$ is H -Galois.

Conversely, if $\text{can}_Q : A \otimes_{A^{\text{co}} Q} A \rightarrow A \otimes Q$ is a bijection, then by Corollary 4.3.4 it is a closed element. By Theorem 4.2.4 we see that $Q = H/K^+H$ for some $K \in \text{Sub}_{\text{gen}}(H)$. Thus Q is a closed element in the Galois connection 4.4. By Theorem 4.6.2(i) the map $\text{can}_Q : H \otimes_{H^{\text{co}} Q} H \rightarrow H \otimes Q$ is an isomorphism. By the commutativity of the above diagram it follows that $\delta_A \otimes \delta$ is an injective map. \square

4.7 Galois theory of crossed products

Let us recall the crossed product construction denoted by $B \#_{\sigma} H$ and introduced in Example 1.3.11(iii). We describe closed elements of the Galois connection (4.5) when A is a crossed product.

Let us note that the results presented below apply to finite Hopf–Galois extensions of division rings, since these are always crossed products by [Montgomery, 1993, Thm 8.3.7].

Theorem 4.7.1

Let $A = B \#_{\sigma} H$ be an H -crossed product over a ring \mathbf{R} such that B is a flat Mittag-Leffler R -module and H is flat as an R -module. Then the Galois correspondence (4.5) exists. Moreover, let us assume that the unit morphism $1_A : R \rightarrow A$, $r \mapsto r1_A$ is a pure homomorphism. Then an element $Q \in \text{Quot}_{\text{gen}}(H)$ is closed if and only if the extension A/A^{coQ} is Q -Galois.

Proof: First of all, the Galois connection (4.5) exists, since we have a diagram:

$$\text{Sub}_{\text{Alg}}(B \#_{\sigma} H / B) \begin{matrix} \xleftarrow{\omega} \\ \xrightarrow{\zeta} \end{matrix} \text{Sub}_{\text{gen}}(H) \begin{matrix} \xleftarrow{\psi} \\ \xrightarrow{\phi} \end{matrix} \text{Quot}_{\text{gen}}(H)$$

where (ϕ, ψ) is the Galois connection (4.4) and $\zeta(S) = B \otimes S$. The poset $\text{Sub}_{\text{gen}}(H)$ is complete, since it is closed under arbitrary joins: for a family $K_i \in \text{Sub}_{\text{gen}}(H)$ ($i \in I$) the join $\bigvee_{i \in I} K_i \in \text{Sub}_{\text{gen}}(H)$ is equal to the subalgebra generated by $\sum_{i \in I} K_i$. The map ζ preserves all intersections. Thus it has a left adjoint ω , i.e. an order preserving map $\omega : \text{Sub}_{\text{Alg}}(B \#_{\sigma} H / B) \rightarrow \text{Sub}_{\text{gen}}(H)$ such that:

$$\omega(S) \subseteq K \iff S \subseteq \zeta(K)$$

for $S \in \text{Sub}_{\text{Alg}}(B \#_{\sigma} H)$ and $K \in \text{Sub}_{\text{gen}}(H)$. Then the Galois connection (4.5) has the form $(\zeta \circ \phi, \psi \circ \omega)$, since by flatness of B we have $B \otimes H^{coQ} = (B \otimes H)^{coQ}$.

Let $Q \in \text{Quot}_{\text{gen}}(H)$ be a closed element of this Galois connection, i.e. $Q = \psi \omega \zeta \phi(Q)$. Then it is closed element of (4.4), since it belongs to the image of ψ . Thus, by Theorem 4.6.2(i), the map: $\text{can}'_Q : H \otimes_{H^{coQ}} H \rightarrow H \otimes Q$, $h \otimes k \mapsto h k_{(1)} \otimes k_{(1)}$ is an isomorphism. We have a commutative diagram:

$$\begin{array}{ccc} A \otimes_{A^{coQ}} A & \xrightarrow{\text{can}_Q} & A \otimes Q = B \#_{\sigma} H \otimes Q \\ \beta \downarrow & \nearrow & \\ B \otimes (H \otimes_{H^{coQ}} H) & & \\ \gamma \downarrow & \nearrow \text{id}_B \otimes \text{can}'_Q & \\ B \otimes (H \otimes_{H^{coQ}} H) & & \end{array} \quad (4.18)$$

where $\beta : (B \#_{\sigma} H) \otimes (B \#_{\sigma} H) \rightarrow B \otimes (H \otimes_{H^{coQ}} H)$ is defined by $\beta(b \#_{\sigma} h \otimes b' \#_{\sigma} h') = (b \#_{\sigma} h \cdot b' \#_{\sigma} 1_H) \otimes h'$ is an isomorphism with the inverse $\beta^{-1}(b \otimes h \otimes k) = (b \#_{\sigma} h) \otimes (1_B \#_{\sigma} k)$; and $\gamma : B \otimes (H \otimes_{H^{coQ}} H) \rightarrow B \otimes (H \otimes_{H^{coQ}} H)$ is defined by $\gamma(b \otimes (h \otimes_{H^{coQ}} k)) = b \sigma(h_{(1)}, k_{(1)}) \otimes (h_{(2)} \otimes_{H^{coQ}} k_{(2)})$. Note

that γ is well defined since it equal to the composition $id_B \otimes (can'_Q)^{-1} \circ can_Q \circ \beta^{-1}$. Observe that γ is an isomorphism with inverse $\gamma^{-1}(b \otimes (h \otimes_{H^{co}Q} k)) = b\sigma^{-1}(h_{(1)}, k_{(1)}) \otimes (h_{(2)} \otimes_{H^{co}Q} k_{(2)})$, which is well defined since $A \#_{\sigma^{-1}} H$ is a crossed product, though it might be nonassociative and nonunital, but most importantly it is a comodule algebra. It follows that can_Q is an isomorphism. The converse follows from Corollary 4.3.4, since $can : A \otimes_B A \rightarrow A \otimes H$ is a bijection. \square

Now we formulate a criterion for closedness of subextensions of A/B which generalises 4.6.2(ii).

Theorem 4.7.2

Let A/B be an H -crossed product extension over a ring \mathbf{R} with B a faithfully flat Mittag-Leffler module and H a flat (thus faithfully flat) \mathbf{R} -module. Then a subalgebra $S \in \text{Sub}_{Alg}(A/B)$ is a closed element of the Galois connection (4.5) if and only if the canonical map:

$$can_S : S \otimes_B A \rightarrow A \square_{\psi(S)} H, \quad can_S(a \otimes_B b) = ab_{(1)} \otimes b_{(2)} \quad (4.19)$$

is an isomorphism.

Proof: First let us note that the map can_S is well defined since it is a composition of $S \otimes_B A \rightarrow A^{co\psi(S)} \otimes_B A$, induced by the inclusion $S \subseteq A^{co\psi(S)}$, with $A^{co\psi(S)} \otimes_B A \rightarrow A \square_{\psi(S)} H$ of [Schneider, 1992b, Thm 1.4].

Let us assume that can_S is an isomorphism. We let $K = H^{co\psi(S)}$. Since B is a flat \mathbf{R} -module we have $(B \#_{\sigma} H)^{co\psi(S)} = B \#_{\sigma} K$. The following diagram commutes:

$$\begin{array}{ccc} S \otimes_B A & \xrightarrow{can_S} & A \square_{\psi(S)} H \\ \alpha \downarrow & & \downarrow \wr \\ B \#_{\sigma} K \otimes H & \xrightarrow{\beta} & B \otimes (H \square_{\psi(S)} H) \end{array} \quad (4.20)$$

where $\alpha : S \otimes_B A \rightarrow (B \#_{\sigma} K) \otimes_B (B \#_{\sigma} H) \rightarrow B \#_{\sigma} K \otimes H$ is given by

$$\alpha(a \otimes_B b \#_{\sigma} h) = (a \cdot b \#_{\sigma} 1_H) \otimes h \quad \text{for } a \in S \subseteq B \#_{\sigma} K, b \#_{\sigma} h \in A$$

It is well defined since K is a left comodule subalgebra of H . The second vertical map $(B \#_{\sigma} H) \square_{\psi(S)} H \rightarrow B \otimes (H \square_{\psi(S)} H)$ is the natural isomorphism, since B is flat over \mathbf{R} . The map $\beta : B \#_{\sigma} K \otimes H \rightarrow B \otimes (H \square_{\psi(S)} H)$ is defined by $\beta(a \#_{\sigma} k \otimes h) = (a \#_{\sigma} k \cdot 1_{B \#_{\sigma} H_{(1)}}) \otimes h_{(2)}$. Note that $\beta = id_B \otimes can_K \circ \gamma$ where $\gamma : (B \#_{\sigma} K) \otimes H \rightarrow (B \#_{\sigma} K) \otimes H$, $\gamma(a \#_{\sigma} k \otimes h) := a\sigma(k_{(1)}, h_{(1)}) \otimes k_{(2)} \otimes h_{(2)}$ is an isomorphism, since σ is convolution invertible, while can_K denotes the canonical map (4.17). Furthermore, the map can_K is an isomorphism by Theorem 4.6.2(ii), since $K := H^{co\psi(S)}$. By commutativity of the above diagram it follows that α is an isomorphism. Now, let us consider the commutative diagram:

$$\begin{array}{ccc}
S \otimes_B (B \#_\sigma H) & \xrightarrow{\alpha} & B \#_\sigma K \otimes H \\
\downarrow & \nearrow \simeq & \\
(B \#_\sigma K) \otimes_B (B \#_\sigma H) & &
\end{array}$$

Thus $S = B \#_\sigma K$ is indeed closed, since α is an isomorphism and $B \#_\sigma H$ is a faithfully flat B -module under the assumptions made.

If S is closed then $S = A^{co\psi(S)} = B \#_\sigma (H^{co\psi(S)})$, since B is a flat as an \mathbf{R} -module. One easily checks that α is an isomorphism with inverse:

$$\alpha^{-1}(a \#_\sigma k \otimes h) = (a \#_\sigma k) \otimes_B (1_B \#_\sigma h)$$

The left coideal subalgebra $K = H^{co\psi(S)}$ is a closed element of (4.4) hence by Theorem 4.6.2(ii) can_K is an isomorphism and thus β is an isomorphism. It follows from (4.20) that can_S is an isomorphism as well. \square

Chapter 5

Coring approach

In this chapter we show, how the concept we have presented of Galois Theory includes classical Galois theory in Field Theory, and which part of classical theory can be covered by the theory developed. In the first section we recall T.Maszczyk approach to Galois extensions using corings. He showed that a field extension \mathbb{E}/\mathbb{F} with Galois group G is a Galois extension if and only if there is a (concrete) isomorphism of corings $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \cong \text{Map}(G, \mathbb{E})$. Hence the latter is a Galois coring in the sense of [Brzeziński \[2002\]](#) (see also [Wisbauer \[2005\]](#)). In the next sections we are going to generalise his approach to a noncommutative setting. We will first develop a Galois correspondence between subalgebras of an H -module algebra and H -module subalgebras of H in Section 5.3 (on page 115). Then we will replace the Galois group acting on the field \mathbb{E} with a Hopf algebra action on a domain A and the coring $\text{Map}(G, A)$ with $\text{Hom}_{\mathbf{k}}(H, A)$. For a left H -module domain A , for which the action satisfies the requirements of Definition 5.4.1 (on page 117), we show that $\text{Hom}_{\mathbf{k}}(H, A)$ has a coring structure (see Proposition 5.4.4 on page 118). In Definition 5.4.1 we require that H has a basis of elements which act on A as monomorphisms. There is also a canonical coring homomorphism from the Sweedler coring:

$$can : A \otimes_{A^H} A \longrightarrow \text{Hom}_{\mathbf{k}}(H, A), \quad can(a \otimes_{A^H} b) = (H \ni h \mapsto a(h \cdot b) \in A) \quad (5.1)$$

where $A^H := \{a \in H : \forall_{h \in H} h \cdot a = \epsilon(h)a\}$. Then we construct a Galois connection between subalgebras of A and quotient corings of $\text{Hom}_{\mathbf{k}}(H, A)$, which extends the Galois connection between subalgebras of A and generalised subalgebras of H (cf. Proposition 5.4.7 on page 121) via: $\text{Sub}_{gen}(H) \ni K \mapsto \text{Hom}_{\mathbf{k}}(K, A) \in \text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$, where $\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ is a complete lattice of quotient corings. Interestingly, this extension gives rise to a bijection between closed subalgebras of H and closed quotient corings of $\text{Hom}_{\mathbf{k}}(H, A)$ (see Corollary 5.4.8 on page 122). Furthermore, if we assume that the above canonical map (5.1) is an epimorphism then a coring $\text{Hom}_{\mathbf{k}}(K, A) \in$

$\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ is a closed element if the canonical map:

$$\text{can} : A \otimes_{A^K} A \longrightarrow \text{Hom}_{\mathbf{k}}(K, A), \quad \text{can}(a \otimes_{A^K} b) = (K \ni k \mapsto a(k \cdot b) \in A)$$

is an isomorphism, where $A^K := \{a \in A : \forall_{k \in K} k \cdot a = \epsilon(k)a\}$. Moreover, if H is a finite dimensional Hopf algebra and A is $\text{Hom}_{\mathbf{k}}(H, A)$ -Galois, i.e. the homomorphism can (5.1) is an isomorphism, then a subalgebra S is closed if the map:

$$\text{can}_S : A \otimes_S A \longrightarrow A \otimes_{A^{\Psi(S)}} A \longrightarrow \text{Hom}(\Psi(S), A)$$

is an isomorphism.

5.1 Classical Galois theory and corings

Here we quote the Maszczyk approach to Galois theory using corings.

Theorem 5.1.1 ([Maszczyk, 2007, Corollary 2.3])

Let $\mathbb{F} \subseteq \mathbb{E}$ be a finite field extension with the Galois group $G = \text{Gal}(\mathbb{E}/\mathbb{F})$. Then $\mathbb{F} \subseteq \mathbb{E}$ is a Galois extension if and only if the map:

$$\begin{aligned} \text{can} : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} &\longrightarrow \text{Map}(G, \mathbb{E}) \\ e_1 \otimes_{\mathbb{F}} e_2 &\longmapsto (g \mapsto e_1 g(e_2)) \end{aligned}$$

is a bijection.

We will omit the proof as it is very similar to the proof of Proposition 1.3.23 (on page 27). Furthermore, in [Maszczyk, 2007, Proposition 2.5] it is shown that can is an isomorphism of corings.

Definition 5.1.2

Let A be a ring (not necessarily commutative). Then an A -coring is a triple $(\mathcal{K}, \Delta, \epsilon)$, where \mathcal{K} is an A -bimodule together with two A -bilinear maps: $\Delta : \mathcal{K} \rightarrow \mathcal{K}_A \otimes_A \mathcal{K}$ and $\epsilon : \mathcal{K} \rightarrow A$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\Delta} & \mathcal{K}_A \otimes_A \mathcal{K} \\ \Delta \downarrow & & \downarrow \Delta_A \otimes_A \text{id}_{\mathcal{K}} \\ \mathcal{K}_A \otimes_A \mathcal{K} & \xrightarrow{\text{id}_{\mathcal{K}_A} \otimes_A \Delta} & \mathcal{K}_A \otimes_A \mathcal{K}_A \otimes_A \mathcal{K} \end{array}$$

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\Delta} & \mathcal{K}_A \otimes_A \mathcal{K} \\ \cong \searrow & & \downarrow \epsilon_A \otimes_A \text{id}_{\mathcal{K}} \\ & & A \otimes_A \mathcal{K} \end{array} \quad \begin{array}{ccc} \mathcal{K} & \xrightarrow{\Delta} & \mathcal{K}_A \otimes_A \mathcal{K} \\ \cong \searrow & & \downarrow \text{id}_{\mathcal{K}_A} \otimes_A \epsilon \\ & & \mathcal{K}_A \otimes_A A \end{array}$$

In other words an A -coring is a comonoid in the monoidal category of A -bimodules. A morphism of corings $f : (\mathcal{K}, \Delta_{\mathcal{K}}, \epsilon_{\mathcal{K}}) \longrightarrow (\mathcal{K}', \Delta_{\mathcal{K}'}, \epsilon_{\mathcal{K}'})$ is an A -bilinear map $f : \mathcal{K} \longrightarrow \mathcal{K}'$ such that: $\Delta_{\mathcal{K}'} \circ f = f \otimes f \circ \Delta_{\mathcal{K}}$ and $\epsilon_{\mathcal{K}'} \circ f = \epsilon_{\mathcal{K}}$.

Example 5.1.3 Here we give two important examples of corings.

1. Let \mathbb{E}/\mathbb{F} be a field extension then $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ is a coring, called the *Sweedler coring*, with the comultiplication:

$$\Delta : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \ni e' \otimes_{\mathbb{F}} e \longmapsto (e \otimes_{\mathbb{F}} 1_{\mathbb{E}}) \otimes_{\mathbb{E}} (1_{\mathbb{E}} \otimes_{\mathbb{F}} e') \in (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E})$$

and counit map $m : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \longrightarrow \mathbb{E}$, the multiplication map of \mathbb{E} .

2. Let \mathbb{E}/\mathbb{F} be a field extension and let G be a finite subgroup of $\text{Gal}(\mathbb{E}/\mathbb{F})$. Then $\text{Map}(G, \mathbb{E})$ is a coring with comultiplication induced by the multiplication of G , $m : G \times G \longrightarrow G$.

$$\Delta : \text{Map}(G, \mathbb{E}) \xrightarrow{m^*} \text{Map}(G \times G, \mathbb{E}) \simeq \text{Map}(G, \mathbb{E}) \otimes_{\mathbb{E}} \text{Map}(G, \mathbb{E})$$

where $\text{Map}(G, \mathbb{E}) \otimes_{\mathbb{E}} \text{Map}(G, \mathbb{E}) \simeq \text{Map}(G \times G, \mathbb{E})$ is given by

$$\phi_1 \otimes_{\mathbb{E}} \phi_2 \mapsto ((g_1, g_2) \mapsto \phi_1(g_1)g_2(\phi_2(g_2)))$$

and the counit $\epsilon : \text{Map}(G, \mathbb{E}) \longrightarrow \mathbb{E}$ given by $\epsilon(\phi) = \phi(e_G)$, where e_G denotes the identity of the group G .

The coring $\text{Map}(G, \mathbb{E})$ was introduced by T. Maszczyk. He observed that the canonical map *can* of Theorem 5.1.1 is a morphism of corings (see [Maszczyk, 2007, Proposition 2.5]).

Proposition 5.1.4

Let $\mathbb{F} \subseteq \mathbb{E}$ be a finite field extension and $G \leq \text{Gal}(\mathbb{E}/\mathbb{F})$ then the map

$$\begin{aligned} \text{can} : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} &\longrightarrow \text{Map}(G, \mathbb{E}) \\ e_1 \otimes_{\mathbb{F}} e_2 &\longmapsto (g \mapsto e_1 g(e_2)) \end{aligned}$$

is a morphism of corings.

Under the assumption that $\mathbb{F} \subseteq \mathbb{E}$ is a finite Galois extension with the Galois group $\text{Gal}(\mathbb{E}/\mathbb{F}) = G$ the coring $\text{Map}(G, \mathbb{E})$ can be realised by many Hopf algebras, i.e. there are many Hopf algebras H such that $\text{Map}(G, \mathbb{E}) \simeq \mathbb{E} \otimes H$ as \mathbb{E} -corings. Let \mathbb{E}/\mathbb{F} be a finite separable field extension. Then Hopf algebras H for which \mathbb{E}/\mathbb{F} is H -Galois are classified as forms of $\tilde{\mathbb{E}}[G]$, where $\tilde{\mathbb{E}}$ is the normal closure of \mathbb{E} and $G := \text{Gal}(\tilde{\mathbb{E}}/\mathbb{F})/\text{Gal}(\tilde{\mathbb{E}}/\mathbb{E})$ (see Greither and Pareigis [1987]).

The canonical map of corings $\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \rightarrow \text{Map}(G, \mathbb{E})$ is closely related to the canonical map of the $\mathbb{K}[G]^*$ -Hopf-Galois extension $\mathbb{F} \subseteq \mathbb{E}$ (where $\mathbb{K} \subseteq \mathbb{F}$ is a finite field extension). The $\mathbb{K}[G]^*$ -comodule structure map $\delta : \mathbb{E} \rightarrow \mathbb{E} \otimes_{\mathbb{K}} \mathbb{K}[G]^*$ is given by

$$\begin{aligned} \mathbb{E} &\longrightarrow \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \xrightarrow{\text{can}} \text{Map}(G, \mathbb{E}) \xrightarrow{\simeq} \mathbb{E} \otimes_{\mathbb{K}} \mathbb{K}[G]^* \\ e &\longmapsto 1 \otimes_{\mathbb{F}} e \longmapsto (g \mapsto g(e)) \longmapsto e_{(0)} \otimes e_{(1)} \end{aligned}$$

where $G = \text{Gal}(\mathbb{E}/\mathbb{F})$ and the last isomorphism is given by

$$\mathbb{E} \otimes_{\mathbb{K}} \mathbb{K}[G]^* \rightarrow \text{Map}(G, \mathbb{E}), \quad e \otimes f \mapsto (G \ni g \mapsto f(g)e \in \mathbb{E})$$

The coaction δ is a \mathbb{K} -linear algebra homomorphism. The composition:

$$\mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \xrightarrow{\text{can}} \text{Map}(G, \mathbb{E}) \cong \mathbb{E} \otimes_{\mathbb{F}} \mathbb{K}[G]^*$$

is the canonical map of \mathbb{E} as a $\mathbb{K}[G]^*$ -comodule algebra. Thus we can realise the isomorphism of corings can as a canonical map of a Hopf-Galois extension. Note that there is no canonical Hopf algebra: \mathbb{K} was any field such that \mathbb{F} is its finite extension. For an infinite group G we cannot use the canonical coring $\text{Map}(G, \mathbb{E})$ but we can still use Hopf algebras.

5.2 Lattices of subcorings and quotient corings

First we focus on the lattices of subcorings and quotient corings. We show that both of them are complete. Then we prove that there is an epimorphism from the lattice of subcorings of the coring $\text{Map}(G, \mathbb{E})$, considered in the previous section, to the lattice of congruences of G when considered as a semigroup. We also show that there is an epimorphism from the lattice of coideals of $\text{Map}(G, \mathbb{E})$ to the lattice of submonoids of G .

Definition 5.2.1

A **subcoring** \mathcal{K}' of an A -coring \mathcal{K} is an A -coring \mathcal{K}' such that \mathcal{K}' is a subbimodule of \mathcal{K} and the inclusion $\mathcal{K}' \subseteq \mathcal{K}$ is a morphism of A -corings.

If \mathcal{K} is pure as a left and right A -module then a subcoring is a subbimodule such that $\Delta|_{\mathcal{K}'}$ takes values in $\mathcal{K}'_A \otimes_A \mathcal{K}' \subseteq \mathcal{K}_A \otimes_A \mathcal{K}$.

Proposition 5.2.2

Subcorings of an A -coring \mathcal{K} form a complete lattice ordered by inclusion.

Proof: Let \mathcal{K}' and \mathcal{K}'' be subcorings with structure maps Δ', ϵ' and Δ'', ϵ'' , respectively. Then $\mathcal{K}' + \mathcal{K}''$ is a subcoring of the coring \mathcal{K} :

$$\begin{array}{ccc}
(\mathcal{K}' \otimes \mathcal{K}') \oplus (\mathcal{K}'' \otimes \mathcal{K}'') & \longrightarrow & (\mathcal{K}' \oplus \mathcal{K}'') \otimes (\mathcal{K}' \oplus \mathcal{K}'') \\
\Delta' \oplus \Delta'' \uparrow & & \downarrow \\
\mathcal{K}' \oplus \mathcal{K}'' & & \\
\downarrow & & \downarrow \\
\mathcal{K}' + \mathcal{K}'' & \xrightarrow{\quad \exists \quad} & (\mathcal{K}' + \mathcal{K}'') \otimes (\mathcal{K}' + \mathcal{K}'') \\
i \downarrow & & \downarrow i \otimes i \\
\mathcal{K} & \xrightarrow{\quad \Delta \quad} & \mathcal{K} \otimes \mathcal{K}
\end{array}$$

The kernel of the composition $f : \mathcal{K}' \oplus \mathcal{K}'' \longrightarrow (\mathcal{K}' + \mathcal{K}'') \otimes (\mathcal{K}' + \mathcal{K}'')$ contains the kernel of the map $\mathcal{K}' \oplus \mathcal{K}'' \longrightarrow \mathcal{K}' + \mathcal{K}''$. Thus the map $\mathcal{K}' + \mathcal{K}'' \longrightarrow (\mathcal{K}' + \mathcal{K}'') \otimes (\mathcal{K}' + \mathcal{K}'')$ exists. Let $(l, -l) \in \ker(\mathcal{K}' \oplus \mathcal{K}'' \longrightarrow \mathcal{K}' + \mathcal{K}'')$ (where $l \in \mathcal{K}' \cap \mathcal{K}''$). Then $f(l, -l) = (l_{(1)'} - l_{(1)'}) \otimes (l_{(2)'} - l_{(2)'})$, where we write $\Delta'(l) = l_{(1)'} \otimes l_{(2)'}$ and $\Delta''(l) = l_{(1)''} \otimes l_{(2)''}$. Then $i \otimes i \circ f(l, -l) = 0$. It remains to observe that the map $i \otimes i : (\mathcal{K}' + \mathcal{K}'') \otimes (\mathcal{K}' + \mathcal{K}'') \longrightarrow \mathcal{K} \otimes \mathcal{K}$ is a monomorphism, which follows from the commutativity of the diagram:

$$\begin{array}{ccc}
\mathcal{K}' + \mathcal{K}'' & \xrightarrow{\Delta' + \Delta''} & (\mathcal{K}' + \mathcal{K}'') \otimes (\mathcal{K}' + \mathcal{K}'') \\
i \downarrow & & \downarrow i \otimes i \\
\mathcal{K} & \xrightarrow{\quad \Delta \quad} & \mathcal{K} \otimes \mathcal{K}
\end{array}$$

□

Definition 5.2.3

Let \mathcal{K} be an A -coring. Then a coideal of \mathcal{K} is a kernel of a morphism of A -corings.

If I is pure as a left and right A -module then a coideal is an A -subbimodule I such that $\Delta(I) \subseteq I_A \otimes_A \mathcal{K} + \mathcal{K}_A \otimes_A I$ and $I \subseteq \ker \epsilon$. Note that by [Brzeziński and Wisbauer, 2003, 17.17] the above definition is equivalent to [Brzeziński and Wisbauer, 2003, 17.14], where coideals are defined as kernels of morphisms of corings which are onto.

Proposition 5.2.4

Let \mathcal{K} be an A -coring. Then the poset $\text{Quot}(\mathcal{K})$ of quotient A -corings is complete.

Proof: It is enough to show that the dual poset of coideals of \mathcal{K} is complete. In order to show this we prove that it is closed under arbitrary suprema. Let $(J_i)_{i \in I}$ be a family of coideals of the coring \mathcal{K} . We let $J := \sum_{i \in I} J_i$, $\pi_i : \mathcal{K} \longrightarrow \mathcal{K}/J_i$ and $\pi : \mathcal{K} \longrightarrow \mathcal{K}/J$ be the natural projections. According to [Brzeziński and Wisbauer, 2003, 17.14] it is enough to show that there exists a coring structure on \mathcal{K}/J . Clearly, we have a diagram:

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\Delta} & \mathcal{K}_A \otimes_A \mathcal{K} \\
\pi \downarrow & & \downarrow \pi \otimes \pi \\
\mathcal{K}/J & \xrightarrow[\overline{\Delta}]{} & \mathcal{K}/J_A \otimes_A \mathcal{K}/J
\end{array}$$

The map $\overline{\Delta}$ exists since for any $\sum_i x_i \in J$, where $x_i \in J_i \forall i \in I$, we have $\pi_i \otimes \pi_i \circ \Delta(x_i) = \Delta_i \circ \pi_i(x_i) = 0$. We get $\pi \otimes \pi \circ \Delta(\sum_i x_i) = 0$ and hence $\ker \pi \subseteq \ker(\pi \otimes \pi \circ \Delta)$. Furthermore, $\overline{\Delta}$ is coassociative. For this we consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{K} & \xrightarrow{\Delta} & \mathcal{K}^{\otimes_A 2} & \xrightleftharpoons[id \otimes \Delta]{\Delta \otimes id} & \mathcal{K}^{\otimes_A 3} \\
\pi \downarrow & & \pi^{\otimes 2} \downarrow & & \downarrow \pi^{\otimes 3} \\
\mathcal{K}/J & \xrightarrow[\overline{\Delta}]{} & \mathcal{K}/J^{\otimes_A 2} & \xrightleftharpoons[id \otimes \overline{\Delta}]{\overline{\Delta} \otimes id} & \mathcal{K}/J^{\otimes_A 3}
\end{array}$$

Since π is an epimorphism and both $\overline{\Delta} \otimes id \circ \overline{\Delta}$ and $id \otimes_A \overline{\Delta} \circ \overline{\Delta}$ makes the outer diagrams commute, they must be equal. The counit is constructed in a similar way:

$$\begin{array}{ccc}
\mathcal{K} & & A \\
\pi \downarrow & \searrow \epsilon & \\
\mathcal{K}/J & \xrightarrow[\bar{\epsilon}]{} & A
\end{array}$$

The counit $\bar{\epsilon}$ exists and is unique such that the above diagram commutes since $\ker \pi \subseteq \ker \epsilon$. The counit axiom can be proved by considering the following diagram:

$$\begin{array}{ccccc}
& & \mathcal{K} & & \\
& \Delta \swarrow & \downarrow & \searrow = & \\
\mathcal{K}^{\otimes_{A^2}} & \xrightarrow{\epsilon \otimes id} & & \xrightarrow{id \otimes \epsilon} & \mathcal{K} \\
& \downarrow & \downarrow & & \downarrow \\
& \bar{\Delta} \swarrow & \mathcal{K}/J & \searrow = & \\
\mathcal{K}/J^{\otimes_{A^2}} & \xrightarrow{\bar{\epsilon} \otimes id} & & \xrightarrow{id \otimes \bar{\epsilon}} & \mathcal{K}/J
\end{array}$$

where all the vertical arrows are the respective projection maps π . From the above diagram we deduce that

$$\begin{aligned}
\bar{\epsilon} \otimes id \circ \bar{\Delta} \circ \pi &= \pi \\
id \otimes \bar{\epsilon} \circ \bar{\Delta} \circ \pi &= \pi
\end{aligned}$$

The map π is an epimorphism, thus $\bar{\epsilon}$ is indeed the counit. Clearly, J is the join of $(J_i)_{i \in I}$ in $\text{cold}(\mathcal{K})$. Hence the poset $\text{cold}(\mathcal{K})$ is complete with respect to the join and thus it is a complete lattice. \square

Proposition 5.2.5

There is a Galois epimorphism from the lattice of subcorings of $\text{Map}(G, \mathbb{E})$ to the lattice of congruences of G , denoted as $\text{Con}(G)$, where G is considered as a semigroup.

As a reminder, a congruence is an equivalence relation which is compatible with all the algebraic operations, in this case, only with multiplication, i.e. if g_1 and h_1 are congruent and g_2 and h_2 are then so is the pair $g_1 g_2$ and $h_1 h_2$, where $g_i, h_i \in G$, $i = 1, 2$. The lattice of congruences of a semigroup is dually isomorphic to the lattice of its quotient semigroups.

Proof: Let C be a subcoring of $\text{Map}(G, \mathbb{E})$. Then we define the corresponding congruence θ_C of G as follows:

$$x \theta_C y \Leftrightarrow \forall_{C \ni f: G \rightarrow \mathbb{E}} f(x) = f(y).$$

For any $\theta \in \text{Con}(G)$ we can associate a subcoring $\text{Map}(G/\theta, \mathbb{E})$. We identify $\text{Map}(G/\theta, \mathbb{E})$ with its image under the map

$$\text{Map}(G/\theta, \mathbb{E}) \longrightarrow \text{Map}(G, \mathbb{E})$$

induced by the quotient map $G \rightarrow G/\theta$. We use the following notation:

$$\begin{array}{ccc} \text{Sub}_{\text{coring}}(\text{Map}(G, \mathbb{E})) & \longrightarrow & \text{Con}(G) \\ C \longmapsto & & \theta_C \\ \\ \text{Con}(G) & \longrightarrow & \text{Sub}_{\text{coring}}(\text{Map}(G, \mathbb{E})) \\ \theta \longmapsto & & C_\theta \end{array}$$

It is straightforward to see that we have just defined antimonotonic morphisms of posets. We check now the Galois epi property: $C \subseteq C_{\theta_C}$ and $\theta = \theta_{C_\theta}$. Let us prove the first property: let $x \in C$ then it is easy to see that one can factorise the map x through G/θ_C so x belongs to $\text{Map}(G/\theta_C, \mathbb{E})$ and thus $C \subseteq \text{Map}(G/\theta_C, \mathbb{E})$. Now we will show that $\theta = \theta_{C_\theta}$. Let us suppose that $x\theta y$. Then for all $f \in C_\theta = \text{Map}(G/\theta, \mathbb{E})$, $f(x) = f(y)$ thus by the definition of θ_{C_θ} we have $x\theta_{C_\theta} y$. Now if $x\theta_{C_\theta} y$ and $x \neg \theta y$ then one can construct a map, as any field \mathbb{E} has at least two elements, from G to \mathbb{E} which factorises through G/θ and such that $f(x) \neq f(y)$. But this contradicts our assumption $x\theta_{C_\theta} y$ so x and y must belong to the same congruence class of θ . \square

Proposition 5.2.6

Let G be a finite group. Then there exists a Galois epimorphism from the lattice of coideals of the coring $\text{Map}(G, \mathbb{E})$ to the lattice of submonoids of G .

Proof: Let I be a coideal of $\text{Map}(G, \mathbb{E})$ then $\bigcap_{f \in I} \ker f$, where $\ker f$ denotes the set of elements of G which are mapped to 0, is a submonoid of G . It contains the identity of the group G , because $I \subseteq \ker \epsilon$, where ϵ is the evaluation at the identity of G . Now let us observe that $\bigcap_{f \in I} \ker f$ is closed under multiplication: if g_1 and g_2 belong to all of the kernels of elements of I then for any $f \in I$ $f(g_1 g_2) = \Delta(f)(g_1, g_2) = 0$, because $\Delta(f) \in I \otimes_{\mathbb{E}} \text{Map}(G, \mathbb{E}) + \text{Map}(G, \mathbb{E}) \otimes_{\mathbb{E}} I$. The map $\theta : I \mapsto \bigcap_{f \in I} \ker f$ reverses the order. The second map of the Galois connection is given as follows. For any submonoid G_0 of G , $\text{Map}(G_0, \mathbb{E})$ forms a coring (the identity element of G is needed to set the counit as evaluation on it). $\text{Map}(G_0, \mathbb{E})$ is a quotient coring of $\text{Map}(G, \mathbb{E})$ via the restriction map $f \mapsto f|_{G_0}$. Thus we have a map ξ from submonoids of G to coideals of the coring $\text{Map}(G, \mathbb{E})$ defined as $\xi(G_0) = \ker(\text{Map}(G, \mathbb{E}) \rightarrow \text{Map}(G_0, \mathbb{E}))$ which reverses the order. Furthermore, one can easily verify that

$$\theta\xi = \text{id}_{\text{Sub}_{\text{mono}}(G)} \quad \text{and} \quad \xi\theta \geq \text{id}_{\text{cold}(\text{Map}(G, \mathbb{E}))}$$

hence θ is an epimorphism which is a part of the Galois connection (θ, ξ) . The second inequality follows since $f \in \xi\theta(I)$ if and only if $\bigcap_{g \in I} \ker g \subseteq \ker f$. \square

In an analogous result for the Hopf algebra $k[G]^*$ we were able to show that every generalised quotient comes from a subgroup of G (Proposition 3.2.10 on page 60). The reason behind this is that right ideals of $k[G]^*$ are always spanned by some of the $\delta_g \in \text{Map}(G, \mathbb{E})$ ($\delta_g(h) = 1$ if and only if $h = g$, otherwise it is 0).

5.3 Galois connection for H -module algebras

We let $A\#H$ denote the smash product of A and H , where A is an H -module algebra (see Example 1.3.113). The underlying vector space of $A\#H$ is $A \otimes H$. We will denote by $a\#h$ (for $a \in A$ and $h \in H$ the simple tensor $a \otimes h$ as an element of the smash product $A\#H$). The multiplication is defined by the rule: $a\#h \cdot b\#k = a(h_{(1)} \cdot b)\#h_{(2)}k$ for $a, b \in A$ and $h, k \in H$. We will denote by $\text{Sub}_{\text{Alg}^H}(H)$ the poset of right coideal subalgebras of H .

Lemma 5.3.1

Let A be an H -module algebra. Then for any $K \in \text{Sub}_{\text{Alg}^H}(H)$ we have:

$$\{a \in A : \forall_{k \in K} k \cdot a = \epsilon(k)a\} = A \cap \text{Cent}_{A\#H}(1_A\#K)$$

where $\text{Cent}_{A\#H}(1_A\#K) = \{x \in A\#H : \forall_{k \in K} x \cdot 1\#k = 1\#k \cdot x\}$ is the centraliser of $1_A\#K$ in $A\#H$.

Proof: The condition $a\#1 \cdot 1\#k = 1\#k \cdot a\#1$ translates to $a\#k = (k_{(1)} \cdot a)\#k_{(2)}$. When we compute $\text{id}_A \otimes \epsilon$ of this equality we get $k \cdot a = \epsilon(k)a$. The other inclusion is obvious. \square

Proposition 5.3.2

Let A be a left H -module algebra over a field \mathbf{k} and let $B = A^H$. We define two order-reversing morphisms:

$$\Phi : \text{Sub}_{\text{Alg}^H}(H) \rightarrow \text{Sub}_{\text{Alg}}(A/B), \Phi(K) := A^K = \{a \in A : \forall_{k \in K} k \cdot a = \epsilon(k)a\}$$

for $K \in \text{Sub}_{\text{Alg}^H}(H)$ a right coideal subalgebra and

$$\Psi : \text{Sub}_{\text{Alg}}(A/B) \rightarrow \text{Sub}_{\text{Alg}^H}(H), \Psi(S) := H \cap \text{Cent}_{A\#H}(S\#1_H)$$

for $S \in \text{Sub}_{\text{Alg}}(A/B)$. Then (Φ, Ψ) is a Galois connection.

Let us note that $\Phi(K)$ is the largest subalgebra of A such that the K -action is left $\Phi(K)$ -linear.

Proof: Let $k \in H$ and $a \in A$. We have

$$\Phi(K) = \{a \in A : \forall_{k \in K} a\#1 \cdot 1\#k = 1\#k \cdot a\#1\}$$

by Lemma 5.3.1. It is clear that $\Phi(K)$ is a subalgebra of A . Let us show that for $S \in \text{Sub}_{\text{Alg}}(A/B)$, $\Psi(S)$ is a right coideal subalgebra of H . From the definition of Ψ it follows that $\Psi(S)$ is a subalgebra of H . Now, for $k \in \Psi(S)$ and for all $a \in S$ we have $(k_{(1)} \cdot a)\#k_{(2)} = a\#k$. Applying $\text{id}_A \otimes \Delta_H$ we get $(k_{(1)(1)} \cdot a)\#k_{(1)(2)} \otimes k_{(2)} = a\#k_{(1)} \otimes k_{(2)}$. It follows that $\Delta_H(k) \in \Psi(S) \otimes H$. Thus K is a right coideal subalgebra of H .

Clearly Φ and Ψ are anti-monotone maps. The Galois properties $\Psi\Phi \geq id_{\text{Sub}_{\text{Hopf}}(H)}$ and $\Phi\Psi \geq id_{\text{Sub}_{\text{Alg}}(A)}$ are easily verified. \square

Note that $\Phi(K)$ is a K -submodule of A :

$$k' \cdot (k \cdot a) = (k'k) \cdot a = \epsilon(k'k)a = \epsilon(k')\epsilon(k)a = \epsilon(k')k \cdot a$$

for $k, k' \in K$ and $a \in \Phi(K)$. Furthermore, if K is a normal Hopf subalgebra then $\Phi(K)$ is an H -submodule:

$$k \cdot (h \cdot a) = (h_{(1)}S(h_{(2)})kh_{(3)}) \cdot a = (h_{(1)}\epsilon(S(h_{(2)})kh_{(3)})) \cdot a = \epsilon(k)h \cdot a$$

for any $k \in K$ and $h \in H$. Let us note that if K is commutative, A^K is an H -submodule, and if moreover A is a faithful H -module then K is one dimensional. For every $h \in H, k \in K$ and $a \in A^K$ we have $k \cdot (h \cdot a) = \epsilon(k)h \cdot a$. The action is faithful so we must have $kh = \epsilon(k)h$. Then for any non-zero $k, k' \in K$ we have $0 \neq kk' = \epsilon(k)k' = \epsilon(k')k$ (it is non zero since the action is faithful).

Let V be a \mathbf{k} -vector space and let $W \subseteq V$ then $W^\perp := \{f \in V^* : f|_W = 0\}$. We also will use the same notation for $W \subseteq V^*$, $W^\perp := \bigcap_{f \in W} \ker f$. Since we will use these two maps only if V is finite dimensional this should not lead to any confusion (under the identification $(V^*)^* = V$ these two maps represent the same morphisms from $\text{Sub}(V^*)$ to $\text{Sub}(V)$).

Let us note that if H is finite dimensional then A is a right H -comodule algebra if and only if A is a left H^* -module algebra. Furthermore, for any $Q \in \text{Quot}_{\text{gen}}(H)$ let I_Q be the right ideal coideal of H such that $Q = H/I_Q$. Then we have $A^{coQ} = A^{I_Q^\perp}$ (I_Q^\perp is a right coideal subalgebra of H^*).

Let (Φ, Ψ) be the Galois connection between $\text{Sub}_{\text{gen}}(H)$ and $\text{Sub}_{\text{Alg}}(A)$, where $\Phi(K) = A^K$. It exists since $\bigvee_{\alpha \in I} K_\alpha$ is equal to the algebra generated by all the K_α ($\alpha \in I$) and thus $A^{\bigvee_{\alpha \in I} K_\alpha} = \bigcap_{\alpha \in I} A^{K_\alpha}$. Let us note that $\phi = \Phi \circ \alpha$, where $\alpha : \text{Quot}_{\text{gen}}(H) \rightarrow \text{Sub}_{\text{Alg}^H}(H^*)$, $\alpha(Q) := Q^*$.

Lemma 5.3.3

Let A be an H -comodule algebra over a field \mathbb{K} , with H a finite dimensional Hopf algebra. Let us also assume that ϕ (and thus Φ) is injective (this holds for example if A is H -Galois). Let $S \in \text{Sub}_{\text{Alg}}(A)$. Then $I_{\psi(S)} = \Psi(S)^\perp$.

Proof: We have the following diagram:

$$\begin{array}{ccc} \text{Sub}_{\text{Alg}}(A) & \xrightleftharpoons[\phi]{\psi} & \text{Quot}_{\text{gen}}(H) \\ & \searrow \Phi & \downarrow \cong \alpha \\ & & \text{Sub}_{\text{Alg}^H}(H^*) \end{array}$$

Ψ (arrow from $\text{Sub}_{\text{Alg}}(A)$ to $\text{Sub}_{\text{Alg}^H}(H^*)$)

where $\alpha(Q) = I_Q^\perp$ is a poset isomorphism. Since $\phi = \Phi \circ \alpha$ we also have $\Psi = \alpha \circ \psi$ and thus $I_{\psi(S)} = \Psi(S)^\perp$. \square

5.4 Maszczyk's approach in a noncommutative way

In this section all algebras are over a fixed base field \mathbf{k} . Furthermore, throughout this section we assume that both H and A are finite dimensional over \mathbf{k} .

Definition 5.4.1

Let A be an H -module algebra with H finite dimensional. We say that H **acts through monomorphisms** if there exists a basis of H , $\{h_i \in H : i \in I\}$ for which the following implication holds: $h_i \cdot a = 0 \Rightarrow a = 0$ for all $i \in I$ and $a \in A$.

Note that the definition requires that $\epsilon(h_i) \neq 0$ by letting $a = 1$. For example if G is a group acting by automorphisms on an algebra A , then it induces a $\mathbf{k}[G]$ -action through monomorphisms on A . Let us note the following theorem:

Theorem 5.4.2 ([Montgomery, 1993, Thm 8.3.7])

Let A be a left H -module algebra, where A is a division ring and H is finite dimensional. Then the following conditions are equivalent:

1. A/A^H is H^* -Galois,
2. $[A : A^H]_r = \dim H$ or $[A : A^H]_l = \dim H$,
3. $A^H \subseteq A$ has the normal basis property,
4. $A \cong A^H \#_\sigma H^*$ is a crossed product (see Example 1.3.11(iii)),
5. $A \# H$ is simple.

An H -action on an algebra A , which has a normal basis, satisfies the requirements of Definition 5.4.1 if there exists a basis $\{h_i \in H : i \in I\}$ of H such that $Hh_i = H$ for all $i \in I$, since the action is induced from the right coaction of H^* on H .

Lemma 5.4.3

Let A be a domain and an H -module algebra, such that the H -action is through monomorphisms (Definition 5.4.1). Let $\phi_1, \phi_2 : H \rightarrow A$ be linear maps. If for all $h, k \in H$, $\phi_1(h_{(1)})h_{(2)} \cdot \phi_2(k) = 0$ then $\phi_1 = 0$ or $\phi_2 = 0$.

Proof: Assume that there exists $k \in H$ such that $\phi_2(k) \neq 0$. Then for every $h \in H$, $\phi_1(h_{(1)}) \otimes h_{(2)} = 0$, since otherwise $\phi_1(h_{(1)})h_{(2)} \cdot \phi_2(k) \neq 0$ (note that we can always write $h_{(1)} \otimes h_{(2)} = \sum_{i \in I} k_i \otimes h_i$ where $h_i, k_i \in H$ and $\{h_i : i \in I\}$ is a basis of elements guaranteed by Definition 5.4.1). It follows that for every

$$h \in H, \phi_1(h) = \phi_1(h_{(1)})\epsilon(h_{(2)}) = 0. \quad \square$$

Proposition 5.4.4

Let A be a domain and an H -module algebra such that H -acts through monomorphisms. Then $\text{Hom}_{\mathbb{K}}(H, A)$ is an A -coring. The A -bimodule structure is given by:

$$\begin{aligned} (a \cdot \phi)(h) &= a\phi(h) \\ (\phi \cdot a)(h) &= \phi(h_{(1)})(h_{(2)} \cdot a) \end{aligned}$$

The comultiplication is given by the commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathbb{K}}(H, A) & \xrightarrow{\Delta} & \text{Hom}_{\mathbb{K}}(H, A) \otimes_A \text{Hom}_{\mathbb{K}}(H, A) \\ & \searrow & \swarrow \alpha \\ & \text{Hom}_{\mathbb{K}}(H \otimes H, A) & \end{array}$$

$\text{Hom}_{\mathbb{K}}(m, A)$

where the isomorphism $\alpha : \text{Hom}_{\mathbb{K}}(H, A) \otimes_A \text{Hom}_{\mathbb{K}}(H, A) \rightarrow \text{Hom}_{\mathbb{K}}(H \otimes H, A)$ is given by: $\alpha(\phi_1 \otimes_A \phi_2) = (h \otimes k \mapsto \phi_1(h_{(1)})h_{(2)} \cdot \phi_2(k))$. The counit is given by $\epsilon(\phi) = \phi(1_H)$.

Proof: The right A module structure is associative since $\text{Hom}_{\mathbb{K}}(H, A)$ is an H -module algebra:

$$\begin{aligned} ((\phi \cdot a) \cdot b)(h) &= (\phi \cdot a)(h_{(1)})h_{(2)} \cdot b \\ &= \phi(h_{(1)})(h_{(2)} \cdot a)(h_{(3)} \cdot b) \\ &= \phi(h_{(1)})h_{(2)} \cdot (ab) \\ &= (\phi \cdot (ab))(h) \end{aligned}$$

for any $a, b \in A$ and $h \in H$. Clearly $\text{Hom}_{\mathbb{K}}(H, A)$ is a left and right A -module. It is an A -bimodule, since both A module structures commute.

Now, let us note that, by Lemma 5.4.3, α is a monomorphism. It is an isomorphism since the dimension of both domain and codomain is $(\dim_{\mathbb{K}} H)^2 \cdot \dim_{\mathbb{K}} A < \infty$. We show that Δ is coassociative. For this we consider the following diagram:

$$\begin{array}{ccccc}
& & \text{Hom}_{\mathbf{k}}(H^{\otimes 2}, A) \otimes_A \text{Hom}_{\mathbf{k}}(H, A) & & \\
& \nearrow \text{Hom}_{\mathbf{k}}(m, A) \otimes_A id & \uparrow \alpha \otimes_A id & \searrow \beta_1 & \\
\text{Hom}_{\mathbf{k}}(H, A) & \xrightarrow{\Delta} & \text{Hom}_{\mathbf{k}}(H, A)^{\otimes_{A^2}} & \xrightarrow{\Delta \otimes_A id} & \text{Hom}_{\mathbf{k}}(H, A)^{\otimes_{A^3}} & \xrightarrow{\beta_1} & \text{Hom}_{\mathbf{k}}(H^{\otimes 3}, A) \\
& \searrow id \otimes_A \text{Hom}_{\mathbf{k}}(m, A) & \downarrow id \otimes_A \alpha & \nearrow \beta_2 & \\
& & \text{Hom}_{\mathbf{k}}(H, A) \otimes_A \text{Hom}_{\mathbf{k}}(H^{\otimes 2}, A) & &
\end{array}$$

where $\beta_1(\Phi \otimes_A \psi)(h \otimes k \otimes l) = \Phi(h_{(1)} \otimes k_{(1)})((h_{(2)}k_{(2)}) \cdot \psi(l))$, and $\beta_2(\phi \otimes_A \Psi)(h \otimes k \otimes l) = \phi(h_{(1)})(h_{(2)} \cdot \Psi(k \otimes l))$. Both maps β_1 and β_2 are monomorphisms, which follows using the same argument as in the proof of Lemma 5.4.3. Thus both $\beta_1 \circ \alpha \otimes_A id$ and $\beta_2 \circ id \otimes_A \alpha$ are monomorphisms. Now, Δ is coassociative if the following equality holds:

$$\beta_1 \circ \text{Hom}_{\mathbf{k}}(m, A) \otimes_A id \circ \Delta = \beta_2 \circ id \otimes_A \text{Hom}_{\mathbf{k}}(m, A)$$

By definition of Δ we have:

$$\phi_{(1)}(h_{(1)})(h_{(2)} \cdot \phi_{(2)}(k)) = \phi(hk) \quad (5.2)$$

We claim that

$$(\beta_1 \circ \text{Hom}_{\mathbf{k}}(m, A) \otimes_A id \circ \Delta)(\phi)(h \otimes k \otimes l) = \phi(hkl)$$

This follows from commutativity of the following diagram:

$$\begin{array}{ccccc}
& & \text{Hom}_{\mathbf{k}}(m \otimes id, A) & & \text{Hom}_{\mathbf{k}}(H^{\otimes 3}, A) \\
& \nearrow \text{Hom}_{\mathbf{k}}(m, A) & \nearrow \text{Hom}_{\mathbf{k}}(H^{\otimes 2}, A) & \nearrow \text{Hom}_{\mathbf{k}}(H^{\otimes 2}, A)^{\otimes_{A^2}} & \uparrow \beta_1 \\
& \uparrow \alpha & \uparrow \alpha & \uparrow \alpha \otimes id & \\
\text{Hom}_{\mathbf{k}}(H, A) & \xrightarrow{\Delta} & \text{Hom}_{\mathbf{k}}(H, A)^{\otimes_{A^2}} & \xrightarrow{\Delta \otimes_A id} & \text{Hom}_{\mathbf{k}}(H, A)^{\otimes_{A^3}}
\end{array}$$

The parallelogram diagram commutes since:

$$\begin{array}{ccc}
(h \otimes k \mapsto \phi(h_{(1)}) h_{(2)} \cdot \phi(k)) & \xrightarrow{\text{Hom}_{\mathbf{k}}(m \otimes id, A)} & (h \otimes k \otimes l \mapsto \phi(h_{(1)}) k_{(1)} (h_{(2)} k_{(2)}) \cdot \psi(l)) \\
\uparrow \alpha & & \uparrow \beta_1 \\
\phi \otimes_A \psi & \xrightarrow{\text{Hom}_{\mathbf{k}}(m, A) \otimes_A id} & \phi \circ m \otimes_A \psi
\end{array}$$

In a similar way we compute $(\beta_2 \circ id \otimes_A \text{Hom}_{\mathbf{k}}(m, A) \circ \Delta)(\phi)(h \otimes k \otimes l) = \phi(hkl)$. Thus, indeed, $\beta_1 \circ \text{Hom}_{\mathbf{k}}(m, A) \otimes_A id \circ \Delta = \beta_2 \circ id \otimes_A \text{Hom}_{\mathbf{k}}(m, A) \circ \Delta$, and hence Δ is coassociative. The counit axiom follows from Equation (5.2). \square

Definition 5.4.5

Let A be an H -module algebra, which is a domain and such that H acts through monomorphisms (Definition 5.4.1). If the canonical map:

$$can : A \otimes_{A^H} A \longrightarrow \text{Hom}_{\mathbf{k}}(H, A), \quad can(a \otimes a') = (h \mapsto a(h \cdot a')) \quad (5.3)$$

is a bijection then the extension A/A^H is called $\text{Hom}_{\mathbf{k}}(H, A)$ -Galois.

Proposition 5.4.6

Let A be an H -module algebra, which is a domain and such that H acts through monomorphisms. Then the canonical map (5.3) is a morphism of corings.

Proof: First we show that can is compatible with comultiplication, i.e. that the following diagram commutes:

$$\begin{array}{ccc}
A \otimes_{A^H} A & \xrightarrow{can} & \text{Hom}_{\mathbf{k}}(H, A) \\
\downarrow & & \downarrow \text{Hom}_{\mathbf{k}}(m, A) \\
(A \otimes_{A^H} A) \otimes_A (A \otimes_{A^H} A) & \xrightarrow{can \otimes_A can} & \text{Hom}_{\mathbf{k}}(H^{\otimes 2}, A) \\
& & \uparrow \alpha \\
& & \text{Hom}_{\mathbf{k}}(H, A) \otimes_A \text{Hom}_{\mathbf{k}}(H, A)
\end{array}$$

For this we take $x \otimes y \in A \otimes_{A^H} A$. Then

$$\text{Hom}_{\mathbf{k}}(m, A) \circ can(x \otimes y) = (h \otimes k \mapsto x(hk) \cdot y)$$

while

$$can \otimes_A can((x \otimes 1) \otimes_A (1 \otimes y)) = ((h \mapsto xh \cdot 1) \otimes_A (k \mapsto k \cdot y))$$

and thus $\alpha \circ \text{can} \otimes_A \text{can} ((x \otimes 1) \otimes_A (1 \otimes y)) = (h \otimes k \mapsto x \epsilon(h_{(1)})(h_{(2)}k) \cdot y)$. Thus the equality follows. Moreover, the morphism can is compatible with the counit: $\epsilon \circ \text{can}(x \otimes y) = xy = \epsilon(x \otimes y)$. \square

If K is a right coideal (unital) subalgebra of H then $\text{Hom}_k(K, A)$ is a quotient coring of $\text{Hom}_k(H, A)$. The quotient map is induced by the inclusion $i : K \subseteq H$ and the comultiplication is given by:

$$\begin{array}{ccc} \text{Hom}_k(K, A) & \xrightarrow{\quad \Delta \quad} & \text{Hom}_k(K, A) \otimes_A \text{Hom}_k(K, A) \\ & \searrow & \swarrow \alpha \\ & \text{Hom}_k(K \otimes K, A) & \end{array}$$

$\text{Hom}_k(m, A)$ \searrow $\text{Hom}_k(K \otimes K, A)$ \swarrow $\text{Hom}_k(K, A) \otimes_A \text{Hom}_k(K, A)$

where α is defined by $\alpha(\phi_1 \otimes_A \phi_2)(h \otimes k) = \phi_1(h_{(1)})h_{(2)} \cdot \phi_2(k)$. It is well defined since K is a right coideal and it is an isomorphism by the same argument as the one used to prove Lemma 5.4.3. The counit of $\text{Hom}(K, A)$ is given by $\epsilon(\phi) = \phi(1_H)$ (K is a unital subalgebra hence $1_H \in K$).

Proposition 5.4.7

Let A and H be as above. Then there exists a Galois connection (Θ, Υ) :

$$\begin{array}{ccc} \text{Sub}_{\text{Alg}}(A) & \xrightleftharpoons[\Phi]{\Psi} & \text{Sub}_{\text{gen}}(H) \\ & \searrow \Upsilon & \downarrow \text{Hom}(-, A) \\ & \text{Quot}(\text{Hom}_k(H, A)) & \end{array} \quad (5.4)$$

Θ \swarrow

where $\Upsilon := \text{Hom}_k(-, A) \circ \Psi$.

Proof: The Galois connection (Φ, Ψ) was constructed in Proposition 5.3.2. To prove that there exist Galois connection (Θ, Υ) it is enough to show that $\text{Hom}_k(-, A)$ preserves all infima, since all the posets are complete. First let us note that $\text{Hom}_k(-, A) : K \mapsto \text{Hom}_k(K, A)$ preserves the order: an inclusion $K_1 \subseteq K_2$ induces an epimorphism of corings: $\text{Hom}_k(K_2, A) \twoheadrightarrow \text{Hom}_k(K_1, A)$ which makes the following diagram commute.

$$\begin{array}{ccc} & \text{Hom}_k(H, A) & \\ \swarrow & & \searrow \\ \text{Hom}_k(K_2, A) & \twoheadrightarrow & \text{Hom}_k(K_1, A) \end{array}$$

Let $(K_i)_{i \in I}$ be a family of right ideal subalgebras. Their meet in $\text{Sub}_{\text{gen}}(H)$ is equal to $\bigcap_{i \in I} K_i$ (see the proof of Proposition 3.1.29). We claim that the

following equality holds $\bigwedge_{i \in I} \text{Hom}_{\mathbf{k}}(K_i, A) = \text{Hom}_{\mathbf{k}}(\bigcap_{i \in I} K_i, A)$. If we show that

$$\sum_i \{f \in \text{Hom}_{\mathbf{k}}(H, A) : f|_{K_i} = 0\} = \left\{f \in \text{Hom}_{\mathbf{k}}(H, A) : f|_{\bigcap_i K_i} = 0\right\} \quad (5.5)$$

then we get that the sequence:

$$0 \rightarrow \left\{f \in \text{Hom}_{\mathbf{k}}(H, A) : f|_{\bigcap_i K_i} = 0\right\} \rightarrow \text{Hom}_{\mathbf{k}}(H, A) \rightarrow \bigwedge_i \text{Hom}_{\mathbf{k}}(K_i, A) \rightarrow 0$$

is exact and the claim follows. Then we get that $\text{Hom}_{\mathbf{k}}(-, A) \circ \Psi$ reflects suprema into infima, and hence there exists a Galois connection (Θ, Υ) . Thus it remains to prove (5.5). For this let us observe that an infinite intersection of subspaces K_i ($i \in I$) of a finite dimensional vector space $\text{Hom}_{\mathbf{k}}(H, A)$ is equal to a finite intersection of K_i ($i \in I_0$), where $I_0 \subseteq I$, $|I_0| < \infty$ (any intersection can be computed as an intersection of a chain $K_{i_0} \supsetneq K_{i_1} \supsetneq \dots$ ($I = \{i_0, i_1, \dots\}$, in which every step is a proper inclusion until it stabilises) and this must stabilise after finitely many steps, since the space $\text{Hom}_{\mathbf{k}}(H, A)$ is finite dimensional). Assuming that (5.5) holds for any finite intersections we have:

$$\sum_{i \in I} K_i^\perp \subseteq \left(\bigcap_{i \in I} K_i\right)^\perp = \left(\bigcap_{i \in I_0} K_i\right)^\perp = \sum_{i \in I_0} K_i^\perp \subseteq \sum_{i \in I} K_i^\perp$$

and thus (5.5) follows. To prove (5.5) in the finite case it is enough to consider $I = \{1, 2\}$. The claim follows since both $K_1^\perp + K_2^\perp$ and $(K_1 \cap K_2)^\perp$ are cokernels of the inclusion $(K_1 + K_2)^\perp \rightarrow K_1^\perp \oplus K_2^\perp$, ($f \mapsto f \oplus f$). In both cases the cokernel map sends $f \oplus g$ to $f - g$. It is clear that the sequence

$$0 \rightarrow (K_1 + K_2)^\perp \rightarrow K_1^\perp \oplus K_2^\perp \rightarrow K_1^\perp + K_2^\perp \rightarrow 0$$

is exact. It remains to show that

$$0 \rightarrow (K_1 + K_2)^\perp \rightarrow K_1^\perp \oplus K_2^\perp \rightarrow (K_1 \cap K_2)^\perp \rightarrow 0$$

is exact. Here the difficulty lies in showing that the map $K_1^\perp \oplus K_2^\perp \ni f \oplus g \mapsto f - g \in (K_1 \cap K_2)^\perp$ is an epimorphism. We can write H as a direct sum $A \oplus B_1 \oplus B_2 \oplus C$ where A, B_i, C are subspaces such that $A = K_1 \cap K_2$, $B_i \oplus A = K_i$ and C is the complement of $K_1 + K_2$ in H . Let $f \in (K_1 \cap K_2)^\perp$, and let g be such that $g|_{B_1} = -f|_{B_1}$ and $g|_{A \oplus B_2 \oplus C} = 0$. Then $f = (f + g) - g$ and $f + g \in K_1^\perp$ and $g \in K_2^\perp$. \square

Corollary 5.4.8

Under the assumptions of the previous Proposition, the morphism $\text{Hom}_{\mathbf{k}}(-, A)$ restricts to a bijection from the closed elements of $\text{Sub}_{\text{gen}}(H)$ in (Φ, Ψ) to the closed elements of $\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ in (Θ, Υ) .

Proof: Clearly, $\text{Hom}_{\mathbf{k}}(-, A) : \text{Sub}_{\text{gen}}(H) \rightarrow \text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ is an injective map, which maps closed elements of $\text{Sub}_{\text{gen}}(H)$ (in (Φ, Ψ)) to closed elements of the poset $\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ (in (Θ, Υ)). If $\mathcal{K} \in \text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ is a closed element then there exists $B \in \text{Sub}_{\text{Alg}}(A)$ such that $\mathcal{K} = \Upsilon(B) = \text{Hom}_{\mathbf{k}}(\Psi(B), A)$. Thus $\text{Hom}_{\mathbf{k}}(-, A)$ is indeed a bijection between the sets of closed elements. \square

Proposition 5.4.9

Let A be a domain and an H -module algebra, such that the H -action is through monomorphisms (Definition 5.4.1). Moreover, let us assume that the canonical map (5.3) is an epimorphism. Let $K, K' \in \text{Sub}_{\text{gen}}(H)$ be such that can_K and $\text{can}_{K'}$ are isomorphisms. Then $\text{Hom}_{\mathbf{k}}(K, A) = \text{Hom}_{\mathbf{k}}(K', A)$ whenever $A^K = A^{K'}$.

Let us note that the above proposition holds even if A or H are infinite dimensional over the base field \mathbf{k} .

Proof: Let $i : K \subseteq H$ and $i' : K' \subseteq H$ be the inclusions. The proposition follows from the commutative diagram:

$$\begin{array}{ccccc}
 & & & \text{Hom}_{\mathbf{k}}(K, A) & \\
 & \nearrow \text{can}_K & & \uparrow \text{Hom}_{\mathbf{k}}(i, A) & \\
 A \otimes_{A^K} A & \xleftarrow{\quad} & A \otimes_{A^H} A & \xrightarrow{\text{can}_H} & \text{Hom}_{\mathbf{k}}(H, A) \\
 & \nwarrow \text{can}_{K'} & & \downarrow \text{Hom}_{\mathbf{k}}(i', A) & \\
 & & & \text{Hom}_{\mathbf{k}}(K', A) &
 \end{array}$$

$\begin{array}{ccc} \xleftarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\ \simeq & \simeq & \end{array}$

The map $f = \text{can}_K \circ \text{can}_{K'}^{-1}$ is a map of corings what easily follows from commutativity of the diagram:

$$\begin{array}{ccccc}
& & \text{Hom}_{\mathbf{k}}(H, A) & & \\
& \swarrow \text{Hom}_{\mathbf{k}}(i', A) & \downarrow & \searrow \text{Hom}_{\mathbf{k}}(i, A) & \\
\text{Hom}_{\mathbf{k}}(K', A) & \xleftarrow{\quad} & & \xrightarrow{\quad} & \text{Hom}_{\mathbf{k}}(K, A) \\
& & \downarrow \Delta_{\text{Hom}_{\mathbf{k}}(H, A)} & & \downarrow \Delta_{\text{Hom}_{\mathbf{k}}(K, A)} \\
& & \text{Hom}_{\mathbf{k}}(H, A) \otimes_A \text{Hom}_{\mathbf{k}}(H, A) & & \\
\downarrow \Delta_{\text{Hom}_{\mathbf{k}}(K', A)} & \swarrow & & \searrow & \downarrow \Delta_{\text{Hom}_{\mathbf{k}}(K, A)} \\
\text{Hom}_{\mathbf{k}}(i', A) \otimes \text{Hom}_{\mathbf{k}}(i', A) & & & & \text{Hom}_{\mathbf{k}}(i, A) \otimes \text{Hom}_{\mathbf{k}}(i, A) \\
& \swarrow & & \searrow & \\
\text{Hom}_{\mathbf{k}}(K', A) \otimes_A \text{Hom}_{\mathbf{k}}(K', A) & \xrightarrow{\quad f \otimes f \quad} & & & \text{Hom}_{\mathbf{k}}(K, A) \otimes_A \text{Hom}_{\mathbf{k}}(K, A)
\end{array}$$

In this way we have proved that $\text{Hom}_{\mathbf{k}}(K', A) \geq \text{Hom}_{\mathbf{k}}(K, A)$ in the poset $\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$. In a similar way we get $\text{Hom}_{\mathbf{k}}(K, A) \geq \text{Hom}_{\mathbf{k}}(K', A)$ and thus they are equal. \square

Corollary 5.4.10

Let H and A be as in Proposition 5.4.9. Let $K \in \text{Sub}_{\text{gen}}(H)$ be such that can_K is an isomorphism. Then $\text{Hom}_{\mathbf{k}}(K, A)$ is a closed element of $\text{Quot}(\text{Hom}_{\mathbf{k}}(H, A))$ in (Θ, Ψ) and thus, by Corollary 5.4.8, K is a closed element of the lattice $\text{Sub}_{\text{gen}}(H)$ in (Φ, Ψ) .

Proof: Let $\tilde{K} = \Psi \circ \Theta(\text{Hom}_{\mathbf{k}}(K, A))$. Then, by Corollary 5.4.8, \tilde{K} is the smallest closed element covering K in $\text{Sub}_{\text{gen}}(K)$ and thus $A^{\tilde{K}} = A^K$. Let $i : K \subseteq \tilde{K}$ be the inclusion. We have the following commutative diagram:

$$\begin{array}{ccc}
A \otimes_{A^H} A & \xrightarrow{\text{can}_H} & \text{Hom}_{\mathbf{k}}(H, A) \\
\downarrow & & \downarrow \\
A \otimes_{A^{\tilde{K}}} A & \xrightarrow{\text{can}_{\tilde{K}}} & \text{Hom}_{\mathbf{k}}(\tilde{K}, A) \\
\downarrow = & & \downarrow \text{Hom}_{\mathbf{k}}(i, A) \\
A \otimes_{A^K} A & \xrightarrow[\text{can}_K]{\cong} & \text{Hom}_{\mathbf{k}}(K, A)
\end{array}$$

From the upper commutative square it follows that $\text{can}_{\tilde{K}}$ is an epimorphism. From the lower commutative square we get that it is also a monomorphism, thus it is an isomorphism. We get that $\text{Hom}_{\mathbf{k}}(i, A)$ is an isomorphism. Thus

$\text{Hom}_{\mathbf{k}}(K, A) = \text{Hom}_{\mathbf{k}}(\tilde{K}, A)$ is closed. \square

Theorem 5.4.11

Let H be a finite dimensional Hopf algebra, A – a domain and an H -module algebra, such that H acts through monomorphisms. Furthermore, let us assume that A is $\text{Hom}_{\mathbf{k}}(H, A)$ -Galois. Then $S \in \text{Sub}_{\text{Alg}}(A)$ is a closed element of the Galois connection (Φ, Ψ) if the map

$$\text{can}_S : A \otimes_S A \rightarrow A \otimes_{A^{\Psi(S)}} A \rightarrow \text{Hom}(\Psi(S), A)$$

is an isomorphism and A is faithfully flat as a right or left S -module. Conversely, if S is closed then can_S is an isomorphism.

Proof: Since H is finite dimensional, A is a right H^* -comodule algebra, and $A^{\text{co}H^*} = A^H$. For $K \in \text{Sub}_{\text{gen}}(H)$ the dual $K^* \in \text{Quot}_{\text{gen}}(H^*)$ and we have a commutative diagram:

$$\begin{array}{ccc} A \otimes_{A^K} A & \xrightarrow{\text{can}_K} & \text{Hom}(K, A) \\ \parallel \downarrow & & \downarrow \parallel \\ A \otimes_{A^{\text{co}K^*}} A & \xrightarrow{\text{can}_{K^*}} & A \otimes K^* \end{array}$$

Since can_H is an isomorphism, can_{H^*} is an isomorphism as well, hence can_{K^*} is an epimorphism and by [Schauenburg and Schneider, 2005, Cor. 3.3] it is an isomorphism. By the above commutative diagram can_K is an isomorphism. Hence, if $S = A^{\Psi(S)}$ then $\text{can}_{\Psi(S)}$ is an isomorphism. Now, if can_S is an isomorphism, then since $\text{can}_{\Psi(S)}$ is an isomorphism we must have $A \otimes_S A = A \otimes_{A^{\Psi(S)}} A$, since A is a domain and by Remark 4.4.3(ii) we have $S = A^{\Psi(S)}$. \square

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